The Use of Partial Stability in the Analysis of Interconnected Systems

Bo Wang
Department of Mechanical Engineering,
Center for Nonlinear Dynamics and Control (CENDAC),
Villanova University,
Villanova, PA 19085
e-mail: bwang6@villanova.edu

Hashem Ashrafiuon
Department of Mechanical Engineering,
Center for Nonlinear Dynamics and Control (CENDAC),
Villanova University,
Villanova, PA 19085
e-mail: hashem.ashrafiuon@villanova.edu

Sergey G. Nersesov[1]
Department of Mechanical Engineering,
Center for Nonlinear Dynamics and Control (CENDAC),
Villanova University,
Villanova, PA 19085
e-mail: sergey.nersesov@villanova.edu

In this paper, we develop sufficient conditions for uniform asymptotic stability of interconnected dynamical systems that are not in cascade form. We show that the stability analysis of a two-subsystem interconnection can be reduced to ensuring the stability of the first nonisolated subsystem with respect to its own state vector (partial stability) and the stability of the isolated second subsystem. In addition, based on the above results, we provide a control design framework for nonlinear systems where the control objective reduces to stabilization of only a part of the system state while guaranteeing the stability for the entire state of the system. We validate the efficacy of the proposed control framework via simulations and experiments using the wheeled mobile robot platform. [DOI: 10.1115/1.4048780]

1 Introduction

Stability of interconnected systems in cascade or hierarchical form has been extensively studied in the literature over the years [1–3]. In particular, for nonlinear systems in cascade given by

$$\dot{x}_1 = f_1(x_1) \quad (1)$$

$$\dot{x}_2 = f_2(x_1, x_2) \quad (2)$$

it was shown in Ref. [4] that global asymptotic stability of Eq. (1) and global asymptotic stability of the isolated subsystem $\dot{x}_2 = f_2(0, x_2)$ guarantee global asymptotic stability of the entire cascade provided that the trajectories of Eq. (2) are bounded. The extensions of these results to time-varying systems in cascade as well as global results have been presented in Ref. [3]. Connections between asymptotic stability of cascaded systems and the input-to-state stability of Eq. (2) have been studied in Refs. [4–6].

More generally, stability of interconnected systems in cascade form containing an arbitrary number of subsystems is presented in Refs. [1] and [2] using traditional Lyapunov techniques. The main approach there is to use stability of the isolated subsystems along with boundedness of the vector field gradient to establish stability of the entire interconnection. The results in Ref. [2] provide the necessary and sufficient conditions for uniform asymptotic stability of the interconnected system. Stability of interconnected systems that are not in cascade form is also a well-studied subject [7,8]. In particular, the passivity theory along with small-gain theorems provides a constructive tool to establish stability of interconnected systems based on the notions of input-to-state stability and input-to-output practical stability [5,6,9]. Alternatively, the notion of vector Lyapunov functions is used in the analysis of large-scale dynamical systems in Ref. [10] while the notion of control vector Lyapunov functions has been introduced in Ref. [11] to provide the means for the decentralized control design of such systems. The results in Refs. [10] and [11] are based on obtaining a Lyapunov function candidate for each individual subsystem and ensuring that the time derivative of this function along the state trajectories of this particular subsystem is less than a specific comparison function. These comparison functions obtained for all subsystems stacked together must form the dynamics of a stable comparison system in order to ensure the stability of the entire interconnected large-scale system. However, it remains an open subject to study stability of such systems based on the approach used for cascade interconnections where stability analysis involves isolated subsystems whose dynamics depend on the individual subsystem states. This is the focus of this work.

In this paper, we combine the notion of partial stability with the stability analysis of interconnected systems that are not in cascade form. Partial stability [12,13] of a dynamical system involves stability of the system with respect to only a part of the system state. In this case, the traditional definitions of Lyapunov, asymptotic, and exponential stability apply to that part of the system state while the rest of the system state has no effect on its stability properties. The main results of this paper provide sufficient conditions for asymptotic stability of a time-varying interconnected dynamical system based on its partial stability with respect to the state of one subsystem and uniform stability of the isolated second subsystem (similar results for exponential and global stability are also developed but omitted here due to page limitation). Furthermore, based on the developed stability results, for controlled nonlinear systems, we present a control design approach for the reference trajectory tracking that requires stabilization of only a part of the system state while guaranteeing the stability of the entire state. In other words, we ensure partial stability of the interconnection through the control design while stability of the second isolated subsystem is guaranteed by the specific design of the error states. Finally, we present an example of a mechanical system where the tracking control is designed using the above framework. The performance of this controller is verified through numerical simulations and experiments.

It should be noted that stability of interconnected systems using the concept of partial stability has been studied in the context of nonlinear observer-based control design [14]. As such, the results of this paper can also be applied to observer-based control design for time-varying nonlinear systems where only partial asymptotic stability of one of the subsystems can be established.

A preliminary version of the results of this paper appeared in Ref. [15]. The additional contribution of this paper includes the control design framework and its application to a mechanical system such as wheeled mobile robot and validation of the controller performance via simulations and experiments.

2 Mathematical Preliminaries

In this section, we establish definitions and notation, review the existing results on stability of interconnected systems, and introduce the notion of partial stability. Let $\mathbb{R}$, $\mathbb{R}_+$, $\overline{\mathbb{R}}_+$ denote the set of real numbers, the set of positive real numbers, and the set of non-negative real numbers, respectively, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $I_n \in \mathbb{R}^{n\times n}$ denote the $n \times n$ identity

[1]Corresponding author.

Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received March 27, 2020; final manuscript received October 7, 2020; published online October 30, 2020. Editor: Ranjan Mukherjee.

matrix, let $||\cdot||$ denote the Euclidean vector norm, and let $V'(t,x) \triangleq [\partial V/\partial x_1, \ldots, \partial V/\partial x_n]$, where $x = [x_1, \ldots, x_n]^T$. We denote an open ball of radius $\delta > 0$ in $\mathbb{R}^n$ as $\mathcal{B}_\delta \triangleq \{x \in \mathbb{R}^n : ||x|| < \delta\}$. Finally, we let $\overset{\circ}{\mathcal{D}}$ denote the interior of the set $\mathcal{D} \subset \mathbb{R}^n$.

Next, we cite one of the main results in Ref. [2] that establishes stability of the interconnected system in cascade form. Specifically, consider a large-scale system $\mathcal{G}$ composed of q interconnected subsystems given by

$$\dot{x}_i(t) = f_i(t, x_1(t), \ldots, x_i(t)), \quad i = 1, \ldots, q \tag{3}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state of the ith subsystem, $\sum_{i=1}^{q} n_i = n$, and $x \triangleq [x_1^T, \ldots, x_q^T]^T \in \mathbb{R}^n$ is the state of $\mathcal{G}$. Associated with Eq. (3), consider a collection of isolated subsystems $\mathcal{G}_i$ given by

$$\dot{x}_i(t) = f_i(t, 0, \ldots, 0, x_i(t)) \tag{4}$$

The following assumptions are needed for the main stability result for the large-scale system $\mathcal{G}$ given by Eq. (3).

Assumption 2.1. *Functions $f_i(\cdot)$ are continuous and*

$$f_i(t, 0, \ldots, 0) = 0, \quad i = 1, \ldots, q, \quad t \geq 0 \tag{5}$$

Assumption 2.2. *There is a constant $c > 0$ such that*

$$\sup_{t\geq 0} \sup_{||w_i|| \leq c} \left\| \frac{\partial f_i(t, w_i)}{\partial w_i} \right\| < \infty, \quad i = 1, \ldots, q \tag{6}$$

where $w_i^T \triangleq [x_1^T, \ldots, x_i^T]$.

The following result presents necessary and sufficient conditions for stability of Eq. (3).

Theorem 2.1 [2]. *Suppose Assumptions 2.1 and 2.2 hold. Then the equilibrium solution $x(t) \equiv 0$ to the large-scale system $\mathcal{G}$ given by Eq. (3) is uniformly asymptotically stable if and only if the equilibrium solution $x_i(t) \equiv 0$ to the isolated subsystem $\mathcal{G}_i$ given by Eq. (4) is uniformly asymptotically stable for all $i = 1, \ldots, q$.*

Next, we review the key definitions and results involving the notion of partial stability [12,13]. Partial stability of a dynamical system deals with the stability with respect to only a part of the system state. Specifically, consider the nonlinear autonomous dynamical system

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \geq 0 \tag{7}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \quad x_2(0) = x_{20} \tag{8}$$

where $x_1 \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^{n_1}$ such that $0 \in \overset{\circ}{\mathcal{D}}$, $x_2 \in \mathbb{R}^{n_2}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$, and $f_1(\cdot, x_2)$ is locally Lipschitz in x_1, $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz in x_2. Note that under the above assumptions, the solution $(x_1(t), x_2(t))$ to (7), (8) exists and is unique over the time interval $\mathcal{I}_{x_0} \subseteq \overline{\mathbb{R}}_+$. Sufficient conditions for existence and uniqueness of solutions over $\mathcal{I}_{x_0} = \overline{\mathbb{R}}_+$ are given in Ref. [16]. The following definition introduces partial Lyapunov and asymptotic stability; that is, stability with respect to x_1, for the nonlinear dynamical system (7) and (8).

Definition 2.1 [13].

(i) *The nonlinear dynamical system (7), (8) is Lyapunov stable with respect to x_1 uniformly in x_{20} if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $||x_{01}|| < \delta$ implies that $||x_1(t)|| < \varepsilon$ for all $t \geq 0$ and for all $x_{20} \in \mathbb{R}^{n_2}$.*

(ii) *The nonlinear dynamical system (7), (8) is asymptotically stable with respect to x_1 uniformly in x_{20} if it is Lyapunov stable with respect to x_1 uniformly in x_{20} and there exists $\delta > 0$ such that $||x_{01}|| < \delta$ implies that $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.*

Next, we present sufficient conditions for the partial stability of Eqs. (7) and (8). For this result, we define $\dot{V}(x_1, x_2) \triangleq = V'(x_1, x_2) f(x_1, x_2)$, where $f(x_1, x_2) \triangleq = [f_1^T(x_1, x_2) f_2^T(x_1, x_2)]^T$, for a given continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$.

Theorem 2.2 [13]. *Consider the nonlinear dynamical system given by Eqs. (7) and (8). Then the following statements hold:*

(1) *If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot)$ such that*

$$\alpha(||x_1||) \leq V(x_1, x_2) \leq \beta(||x_1||), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2} \tag{9}$$

$$\dot{V}(x_1, x_2) \leq 0, \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2} \tag{10}$$

then the nonlinear dynamical system given by (7), (8) is Lyapunov stable with respect to x_1 uniformly in x_{20}.

(2) *If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$ such that (9) holds and*

$$\dot{V}(x_1, x_2) \leq -\gamma(||x_1||), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2} \tag{11}$$

then the nonlinear dynamical system given by (7), (8) is asymptotically stable with respect to x_1 uniformly in x_{20}.

3 Main Result

In this section, we present our main result involving sufficient conditions for stability of interconnected systems that are not in cascade form. For this, consider the interconnected dynamical system consisting of two subsystems given by

$$\dot{x}_1(t) = f_1(t, x_1(t), x_2(t)), \quad x_1(t_0) = x_{10}, \quad t \geq t_0 \tag{12}$$

$$\dot{x}_2(t) = f_2(t, x_1(t), x_2(t)), \quad x_2(t_0) = x_{20} \tag{13}$$

where $t_0 \geq 0$, $x_1 \in \mathcal{D}_1$, $\mathcal{D}_1 \subseteq \mathbb{R}^{n_1}$ such that $0 \in \overset{\circ}{\mathcal{D}}_1$, $x_2 \in \mathcal{D}_2$, $\mathcal{D}_2 \subseteq \mathbb{R}^{n_2}$ such that $0 \in \overset{\circ}{\mathcal{D}}_2$, $f_1 : \overline{\mathbb{R}}_+ \times \mathcal{D}_1 \times \mathcal{D}_2 \to \mathbb{R}^{n_1}$ is such that $f_1(t, 0, x_2) = 0$ for every $t \geq 0$ and $x_2 \in \mathcal{D}_2$, $f_2 : \overline{\mathbb{R}}_+ \times \mathcal{D}_1 \times \mathcal{D}_2 \to \mathbb{R}^{n_2}$ is such that $f_2(t, 0, 0) = 0$ for every $t \geq 0$. We assume that sufficient conditions for existence and uniqueness of solutions to Eqs. (12) and (13) are satisfied [16]. Note that the interconnected system (12), (13) can be written in the form of Eqs. (7) and (8) by introducing a new state $\tilde{x}_2 \triangleq [x_2^T, t]^T$ and the new time variable $\tau = t - t_0$. In this case, Eqs. (12) and (13) are equivalent to

$$\dot{x}_1(\tau) = f_1(x_1(\tau), \tilde{x}_2(\tau)), \quad x_1(0) = x_{10}, \quad \tau \geq 0 \tag{14}$$

$$\dot{\tilde{x}}_2(\tau) = \tilde{f}_2(x_1(\tau), \tilde{x}_2(\tau)), \quad \tilde{x}_2(0) = [x_{20}^T, t_0]^T \tag{15}$$

where $\tilde{f}_2(x_1, \tilde{x}_2) \triangleq [f_2^T(x_1, \tilde{x}_2) \ 1]^T$ and the differentiation is taken with respect to τ. For the main result of this section, we formulate an assumption similar to the Assumption 2.2.

Assumption 3.1. There is a constant $c > 0$ such that

$$\sup_{t\geq 0} \sup_{||x_1|| \leq c, ||x_2|| \leq c} \left| \left| \frac{\partial f_2(t, x_1, x_2)}{\partial x_1} \right| \right| < \infty \tag{16}$$

Theorem 3.1. *Suppose Assumption 3.1 holds. Then the system (12), (13) is uniformly asymptotically stable if the system (14), (15) is asymptotically stable with respect to x_1 uniformly in $\tilde{x}_{20}$ and the system*

$$\dot{x}_2(t) = f_2(t, 0, x_2(t)), \quad x_2(t_0) = x_{20}, \quad t \geq t_0, \quad t_0 \geq 0 \tag{17}$$

is uniformly asymptotically stable.

Proof. For the proof of this result, we use notation $x \triangleq [x_1^T, x_2^T]^T$ and $x_0 \triangleq [x_{10}^T, x_{20}^T]^T$. To show that the system (12), (13) is uniformly Lyapunov stable, we need to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $||x_0|| < \delta$ implies that $||x(t)|| < \varepsilon$ for all $t \geq t_0$ and for all $t_0 \geq 0$. This is equivalent to showing that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $||x_{10}|| < \delta$ and $||x_{20}|| < \delta$ imply $||x_1(t)|| < \varepsilon$ and $||x_2(t)|| < \varepsilon$, $t \geq t_0$, for all $t_0 \geq 0$.

Note that since (17) is uniformly asymptotically stable, it follows from Theorem 4.16 in Ref. [16] that there exists a continuously differentiable function $V_2(t, x_2)$, $x_2 \in \mathcal{B}_c$, $t \geq t_0$, and class $\mathcal{K}$ functions $\psi_1(\cdot)$, $\psi_2(\cdot)$, and $\psi_3(\cdot)$ such that

$$\psi_1(||x_2||) \leq V_2(t, x_2) \leq \psi_2(||x_2||), \quad x_2 \in \mathcal{B}_c, \quad t \geq t_0 \tag{18}$$

along the trajectories of Eq. (17)

$$\begin{aligned}\dot{V}_{2(17)}(t, x_2) &= \frac{\partial}{\partial t} V_2(t, x_2) + V_2'(t, x_2) f_2(t, 0, x_2) \\ &\leq -\psi_3(||x_2||), \quad x_2 \in \mathcal{B}_c, \quad t \geq t_0\end{aligned} \tag{19}$$

and there exists $L > 0$ such that

$$\sup_{t \geq 0} \sup_{||x_2|| \leq c} ||V_2'(t, x_2)|| = L < \infty \tag{20}$$

Also, it follows from Assumption 3.1 that there exists $R > 0$ such that

$$\sup_{t \geq 0} \sup_{||x_1|| \leq c, ||x_2|| \leq c} \left|\left| \frac{\partial f_2(t, x_1, x_2)}{\partial x_1} \right|\right| = R < \infty \tag{21}$$

Next, choose arbitrary $\varepsilon > 0$ and let $\varepsilon_1 > 0$ be such that $\psi_2(\varepsilon_1) < \psi_1(\varepsilon)$. Note that, in this case, $\varepsilon_1 < \varepsilon$. Furthermore, choose $\delta_1 < \varepsilon_1$ such that

$$LR\delta_1 < \psi_3(\varepsilon_1) \tag{22}$$

It follows from asymptotic stability of Eqs. (14) and (15) with respect to x_1 uniformly in $\tilde{x}_{20} \triangleq [x_{20}^T, t_0]^T$ that there exists $\delta > 0$ such that if $||x_{10}|| < \delta$, then $||x_1(t)|| < \delta_1$, $t \geq t_0$, for all $x_{20} \in \mathcal{B}_c$ and $t_0 \geq 0$. Next, consider the time derivative of $V_2(\cdot, \cdot)$ along the trajectories of Eq. (13), that is

$$\begin{aligned}\dot{V}_{2(13)}(t, x_2) &= \frac{\partial}{\partial t} V_2(t, x_2) + V_2'(t, x_2) f_2(t, x_1, x_2) \\ &= \frac{\partial}{\partial t} V_2(t, x_2) + V_2'(t, x_2) f_2(t, 0, x_2) \\ &\quad + V_2'(t, x_2)\left[f_2(t, x_1, x_2) - f_2(t, 0, x_2)\right] \\ &= \dot{V}_{2(17)}(t, x_2) + V_2'(t, x_2)\left[f_2(t, x_1, x_2) - f_2(t, 0, x_2)\right] \\ &\quad t \geq t_0, \quad x_1 \in \mathcal{B}_c, \quad x_2 \in \mathcal{B}_c\end{aligned} \tag{23}$$

It follows from Eq. (19) and the mean value theorem [17] that

$$\begin{aligned}\dot{V}_{2(13)}(t, x_2) &\leq -\psi_3(||x_2||) + LR||x_1|| \\ &\leq -\psi_3(||x_2||) + LR\delta_1, \quad x_2 \in \mathcal{B}_c, \quad t \geq t_0\end{aligned} \tag{24}$$

It can be seen from Eqs. (22) and (24) that

$$\dot{V}_{2(13)}(t, x_2) < 0, \quad \varepsilon_1 \leq ||x_2|| \leq \varepsilon, \quad t \geq t_0 \tag{25}$$

Now, let $||x_{20}|| < \delta$ and assume, *ad absurdum*, that there exists $t^* > t_0$ such that $||x_2(t^*)|| = \varepsilon$. In this case, there must exist $t_1 > t_0$ such that $t_1 < t^*$ and $||x_2(t_1)|| = \varepsilon_1$. Now, it follows from Eqs. (18) and (25) that

$$\begin{aligned}\psi_1(\varepsilon) = \psi_1(||x_2(t^*)||) &\leq V_2(t^*, x_2(t^*)) < V_2(t_1, x_2(t_1)) \\ &\leq \psi_2(||x_2(t_1)||) = \psi_2(\varepsilon_1)\end{aligned} \tag{26}$$

which implies that $\psi_1(\varepsilon) < \psi_2(\varepsilon_1)$. This is a contradiction since ε_1 was chosen such that $\psi_2(\varepsilon_1) < \psi_1(\varepsilon)$. Thus, for the system (12), (13), $||x_{10}|| < \delta$ and $||x_{20}|| < \delta$ imply $||x_1(t)|| < \varepsilon$, $t \geq t_0$, and $||x_2(t)|| < \varepsilon$, $t \geq t_0$, for all $t_0 \geq 0$ which shows uniform Lyapunov stability of Eqs. (12) and (13).

To show uniform convergence for the system (12), (13), we need to show that there exists $\delta > 0$ such that $||x_0|| < \delta$ implies $\lim_{t \to \infty} x(t) = 0$ uniformly in x_0 and t_0 for all $t_0 \geq 0$. This is equivalent to showing that there exists $\delta > 0$ such that $||x_{10}|| < \delta$ and $||x_{20}|| < \delta$ implies $\lim_{t \to \infty} x_1(t) = 0$ and $\lim_{t \to \infty} x_2(t) = 0$ uniformly in x_{10}, x_{20}, and t_0 for all $t_0 > 0$. Note that it follows from asymptotic stability of Eqs. (14) and (15) with respect to x_1 uniformly in $\tilde{x}_{20} = [x_{20}^T, t_0]^T$ that there exists $\delta_1 > 0$ such that for the system (12), (13), $||x_{10}|| < \delta_1$ implies that $\lim_{t \to \infty} x_1(t) = 0$ uniformly in x_{10}, x_{20}, and t_0 for all $x_{20} \in \mathcal{B}_c$ and $t_0 \geq 0$. Also, note that from the uniform asymptotic stability of the system (17), it follows that there exists $\delta_2 > 0$ such that, for the system (17), $||x_{20}|| < \delta_2$ implies $\lim_{t \to \infty} x_2(t) = 0$ uniformly in x_{20} and t_0 for all $t_0 \geq 0$.

Let $\tilde{\delta} = \min\{\delta_1, \delta_2\}$ and it follows from uniform Lyapunov stability of Eqs. (12) and (13) shown above that for $\tilde{\delta}$ there exists $\delta > 0$ such that $||x_{10}|| < \delta$ and $||x_{20}|| < \delta$ imply $||x_1(t)|| < \tilde{\delta}$, $t \geq t_0$, and $||x_2(t)|| < \tilde{\delta}$, $t \geq t_0$, for all $t_0 \geq 0$. In this case, it follows from Eq. (24) that, for all $t \geq t_0$

$$\begin{aligned}\dot{V}_{2(13)}(t, x_2(t)) &\leq -\psi_3(||x_2(t)||) + LR||x_1(t)|| \\ &\leq -\psi_3(\psi_2^{-1}(V_2(t, x_2(t)))) + LR||x_1(t)||\end{aligned} \tag{27}$$

Using comparison principle [16], it can be shown that

$$V_2(t, x_2(t)) \leq V(t - t_0), \quad t \geq t_0 \tag{28}$$

where $V(\cdot)$ is the solution to the differential equation

$$\dot{V}(t) = -\psi_3(\psi_2^{-1}(V(t))) + LR||x_1(t)||, \quad V(0) = V_2(t_0, x_{20}) \tag{29}$$

It was shown in Ref. [1] that $\lim_{t \to \infty} V(t) = 0$ uniformly in x_{10}, x_{20}, and t_0. Hence, it follows that

$$\psi_1(||x_2(t)||) \leq V_2(t, x_2(t)) \leq V(t - t_0), \quad t \geq t_0 \tag{30}$$

which implies that $\lim_{t \to \infty} x_2(t) = 0$ uniformly in x_{10}, x_{20}, and t_0 for all $t_0 \geq 0$. ■

4 Application to Control

The result of Sec. 3 is developed for interconnected dynamical systems. However, this result can also be used in control design applications. To see this, consider a controlled nonlinear dynamical system given by

$$\dot{x}_1(t) = f_1(t, x_1(t), x_2(t)), \quad x_1(t_0) = x_{10}, \quad t \geq t_0 \tag{31}$$

$$\dot{x}_2(t) = f_2(t, x_1(t), x_2(t), u(t)), \quad x_2(t_0) = x_{20} \tag{32}$$

where $t_0 \geq 0$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and $u(\cdot) \in \mathbb{R}^m$ are the control inputs. Note that underactuated mechanical systems are usually represented by Eqs. (31) and (32), where $f_2(\cdot, \cdot, \cdot, \cdot)$ may also contain unknown terms and disturbances. For the main result of this section, we make the following assumption.

Assumption 4.1. *Assume that there exists a mapping* $g : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ *such that* x_2 *can be explicitly expressed as a function of* x_1 *and* $\dot{x}_1$ *given by*

$$x_2 = g(t, x_1, \dot{x}_1) \tag{33}$$

Consider the control objective to design a control law $u(\cdot)$ that ensures that the entire state of Eqs. (31) and (32) asymptotically tracks the desired trajectory, that is, $x_1(t) \to x_{1d}(t)$ and $x_2(t) \to x_{2d}(t)$ as $t \to \infty$, where $x_{1d}(t)$, $x_{2d}(t)$ are given sufficiently smooth functions. Furthermore, we assume that the desired trajectory $(x_{1d}(t), x_{2d}(t))$, $t \geq t_0$ is not arbitrary but obeys the same constraints as the actual system, that is

$$\dot{x}_{1d}(t) = f_1(t, x_{1d}(t), x_{2d}(t)), \quad t \geq t_0 \tag{34}$$

Note that it follows from Assumption 4.1 and (34) that

$$x_{2d}(t) = g(t, x_{1d}(t), \dot{x}_{1d}(t)), \quad t \geq t_0 \tag{35}$$

Next, instead of defining the error states that are simply differences between the actual and the desired states, introduce [18]

$$\mu(t) \triangleq \dot{x}_{1\mathrm{d}}(t) + M(x_1 - x_{1\mathrm{d}}(t)) \tag{36}$$

where $M \in \mathbb{R}^{n_1 \times n_1}$ is Hurwitz, and introduce the following error states

$$e_1 \triangleq \dot{x}_1 - \mu(t), \quad e_2 \triangleq x_1 - x_{1\mathrm{d}}(t) \tag{37}$$

Note that if $e_1(t) \to 0$ and $e_2(t) \to 0$ as $t \to \infty$, then $x_1(t) \to x_{1\mathrm{d}}(t)$ as $t \to \infty$ and it follows from Eqs. (36) and (37) that $\dot{x}_1(t) \to \mu(t) \to \dot{x}_{1\mathrm{d}}(t)$ as $t \to \infty$. This along with Eqs. (33) and (35) implies that $x_2(t) \to x_{2\mathrm{d}}(t)$ as $t \to \infty$. Thus, the fact that $e_1(t) \to 0$ and $e_2(t) \to 0$ as $t \to \infty$ accomplishes the control objective.

Next, based on Eqs. (31), (32), and (37), we develop the error dynamics given by

$$\dot{e}_1(t) = F(t, e_1(t), e_2(t), u(t)) \tag{38}$$

$$\dot{e}_2(t) = Me_2(t) + e_1(t) \tag{39}$$

where

$$\begin{aligned} F(t, e_1, e_2, u(t)) \triangleq & \left(\frac{\partial f_1}{\partial x_1}(t, x_1, x_2) - M\right)[e_1 + Me_2 + \dot{x}_{1\mathrm{d}}(t)] \\ & + M\dot{x}_{1\mathrm{d}}(t) - \ddot{x}_{1\mathrm{d}}(t) \\ & + \frac{\partial f_1}{\partial t}(t, x_1, x_2) \\ & + \frac{\partial f_1}{\partial x_2}(t, x_1, x_2) f_2(t, x_1, x_2, u(t)) \end{aligned} \tag{40}$$

with x_1, $\dot{x}_1$, and x_2 replaced by their functional dependencies on e_1 and e_2 given by

$$x_1 = e_2 + x_{1\mathrm{d}}(t) \tag{41}$$

$$\dot{x}_1 = e_1 + Me_2 + \dot{x}_{1\mathrm{d}}(t) \tag{42}$$

$$x_2 = g(t, e_2 + x_{1\mathrm{d}}(t), e_1 + Me_2 + \dot{x}_{1\mathrm{d}}(t)) \tag{43}$$

Now, note that the dynamics of Eq. (39) are asymptotically stable with $e_1 \equiv 0$ since $M \in \mathbb{R}^{n_1 \times n_1}$ is Hurwitz. Thus, using the result of Theorem 3.1, the control objective reduces to designing $u(\cdot)$ that ensures that (38), (39) is asymptotically stable with respect to e_1 uniformly in e_{20} and t_0. In other words, the control design reduces to stabilization of e_1 only, rather than both e_1 and e_2.

Remark 4.1. Typically, in mechanical systems whose dynamics have the structure of Eqs. (31) and (32), x_1 represents the vector of generalized positions and x_2 represents the vector of generalized velocities. In particular, the motion of a six degree-of-freedom body in the three-dimensional space is characterized by twelve equations known as Euler equations. Specifically, the six kinematic equations provide the relationship between the time derivatives of the Earth-fixed position and orientation variables, $x_1 \in \mathbb{R}^6$, and the body-frame velocities and angular rates, $x_2 \in \mathbb{R}^6$, *while the remaining six equations represent the dependence of* $\dot{x}_2$ *on external forces and moments. These equations have the form*

$$\dot{x}_1(t) = T(x_1(t))x_2(t) \tag{44}$$

$$\dot{x}_2(t) = f_2(t, x_1(t), x_2(t), u(t)) \tag{45}$$

where $T(\cdot) \in \mathbb{R}^{6\times 6}$ is invertible within the feasible range of motion and $u(\cdot)$ is the vector of forces and moments. Hence, from Eq. (44), the state variable x_2 can be explicitly expressed as a function of x_1 and $\dot{x}_1$ given by

$$x_2 = T^{-1}(x_1)\dot{x}_1 \tag{46}$$

This ensures that Eq. (44) satisfies Assumption 4.1 and makes the control design framework developed for Eqs. (31) and (32) applicable to Eqs. (44) and (45).

5 Example

To demonstrate the efficacy of the control design framework presented in Sec. 4, we design a tracking controller for a wheeled mobile robot whose equations of motion are given by

$$\dot{x}(t) = v_x(t)\cos\theta(t) \tag{47}$$

$$\dot{y}(t) = v_x(t)\sin\theta(t) \tag{48}$$

$$\dot{\theta}(t) = \omega(t) \tag{49}$$

$$\dot{v}_x(t) = u_1(t) + \delta_1(v_x(t), \omega(t)) \tag{50}$$

$$\dot{\omega}(t) = u_2(t) + \delta_2(v_x(t), \omega(t)) \tag{51}$$

where (x, y) represent the position of the mass center with respect to the origin of a ground-fixed reference frame, θ is the orientation angle with respect to the ground-fixed x-axis, v_x is the forward speed of the robot's mass center, u_1 and u_2 are generic control inputs that contain known robot dynamics and the torques rotating the wheels, and δ_1, δ_2 are unknown terms and disturbances. Note that once the control algorithms u_1, u_2 are designed, the rotating torques can be uniquely determined (see Ref. [18], for details). Next, we partition the state of Eqs. (47)–(51) into $x_1 \triangleq [x, y]^T$ and $x_2 \triangleq [\theta, v_x, \omega]^T$. Note that the mapping between x_2 and x_1 is given by

$$\theta = \arctan\left(\frac{\dot{y}}{\dot{x}}\right) \tag{52}$$

$$v_x = \dot{x}\cos\left(\arctan\left(\frac{\dot{y}}{\dot{x}}\right)\right) + \dot{y}\sin\left(\arctan\left(\frac{\dot{y}}{\dot{x}}\right)\right) \tag{53}$$

$$\omega = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \tag{54}$$

The objective is to design u_1 and u_2 such that the entire state of Eqs. (47)–(51) asymptotically tracks the desired trajectory, that is, $x_1(t) \to x_{1\mathrm{d}}(t)$ and $x_2(t) \to x_{2\mathrm{d}}(t)$ as $t \to \infty$, where $x_{1\mathrm{d}}(t)$, $x_{2\mathrm{d}}(t)$ are given smooth functions. Now, if we continue and define the error states as in Eq. (37), then the error dynamics will only contain u_1. In order to avoid this and take advantage of two control inputs, we consider a coordinate transformation [19] given by

$$z_1 = x_1 + L\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}, \quad L > 0 \tag{55}$$

$$z_2 = x_2 \tag{56}$$

Note that if $z_1(t) \to z_{1\mathrm{d}}(t)$ and $z_2(t) \to z_{2\mathrm{d}}(t)$ as $t \to \infty$, then $x_1(t) \to x_{1\mathrm{d}}(t)$ and $x_2(t) \to x_{2\mathrm{d}}(t)$ as $t \to \infty$. Next, define

$$\mu(t) = \dot{z}_{1\mathrm{d}}(t) + M(z_1 - z_{1\mathrm{d}}(t)) \tag{57}$$

and the error states

$$e_1 = \dot{z}_1 - \mu(t), \quad e_2 \triangleq z_1 - z_{1\mathrm{d}}(t) \tag{58}$$

Then the error dynamics for e_2 are given by Eq. (39) while the error dynamics for e_1 are given by

$$\dot{e}_1(t) = F(t, e_1, e_2) + B(e_1, e_2)\begin{bmatrix}u_1 \\ u_2\end{bmatrix} + B(e_1, e_2)\tilde{\delta} \tag{59}$$

where

$$\begin{aligned} F(t, e_1, e_2) \triangleq & \begin{bmatrix} -v_x\omega\sin\theta - L\omega^2\cos\theta \\ v_x\omega\cos\theta - L\omega^2\sin\theta \end{bmatrix} \\ & - M\begin{bmatrix} v_x\cos\theta - L\omega\sin\theta \\ v_x\sin\theta + L\omega\cos\theta \end{bmatrix} \\ & + M\dot{z}_{1\mathrm{d}}(t) - \ddot{z}_{1\mathrm{d}}(t) \end{aligned} \tag{60}$$

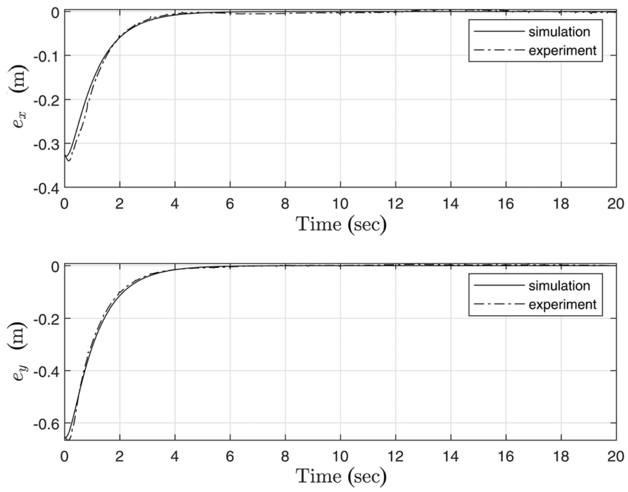

Fig. 1 Error in *x* and *y* variables versus time

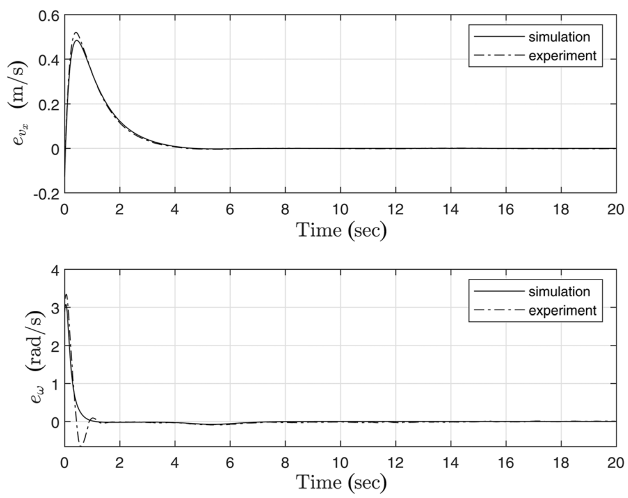

Fig. 2 Error in v_x and ω variables versus time

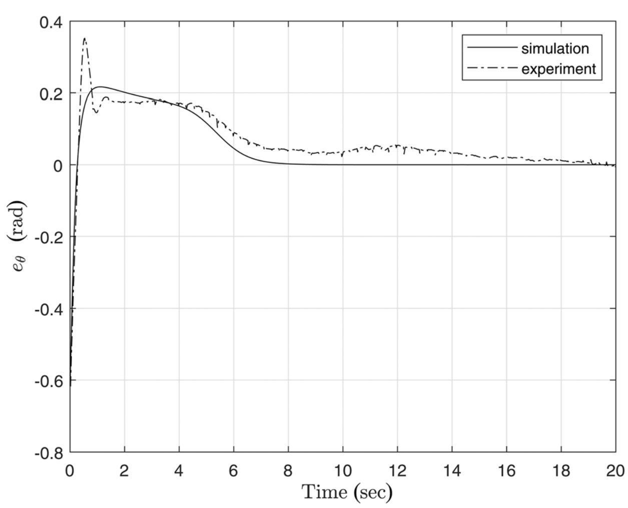

Fig. 3 Error in θ variable versus time

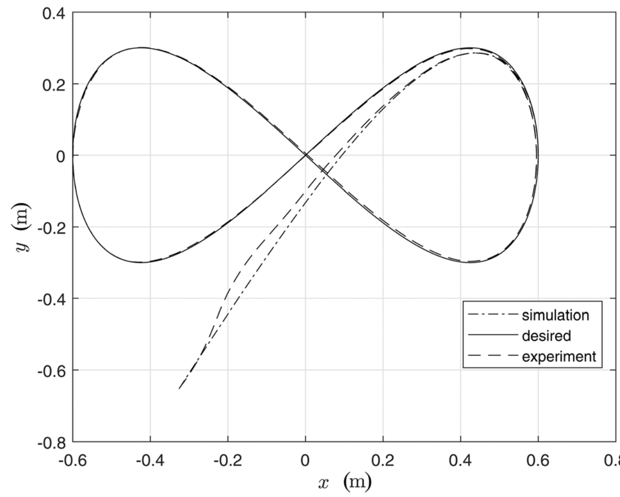

Fig. 4 Desired and actual trajectories

$$B(e_1, e_2) \triangleq \begin{bmatrix} \cos\theta & -L\sin\theta \\ \sin\theta & L\cos\theta \end{bmatrix}$$

$$\tilde{\delta} \triangleq \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{61}$$

Thus, the control task reduces to partial stabilization of Eq. (59) with respect to e_1 only. If $||\tilde{\delta}(e_1, e_2)|| \leq \gamma||e_1||$ for some $\gamma > 0$, then a simple feedback linearizing controller given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = B^{-1}(e_1, e_2)(-F(t, e_1, e_2) + Ae_1) \tag{62}$$

where $A \in \mathbb{R}^{2\times 2}$ is Hurwitz and $\lambda_{\min}(A^T + A) < -2\gamma$ will uniformly asymptotically stabilize (59) with respect to e_1.

6 Validation

We validated performance of the controller (62) with both simulations and experimental runs using wheeled mobile robot in our laboratory setup. The robot mass and geometric properties can be found in Ref. [18]. The results below are presented for a "figure eight" trajectory given by $x_d(t) = 0.6\sin(0.2t)$, $t \geq 0$, $y_d(t) = 0.3\sin(0.4t)$, $t \geq 0$, and with the controller parameters given by $L = 0.4$, $M = -I_2$, and $A = -5I_2$. Figures 1–3 show the difference between the actual and desired variables versus time for simulation and experiment while Fig. 4 shows the desired and actual trajectories. We also validated performance of the controller (62) for a straight line trajectory but due to page limitation we do not present these results here.

7 Conclusion

In this paper, we developed sufficient conditions for uniform asymptotic stability of noncascade interconnected systems based on the partial stability of the first subsystem with respect to its own state and the uniform asymptotic stability of the isolated second subsystem. Moreover, we applied this stability analysis framework to the control design problem where the control objective reduces to stabilization of only a part of the system state while guaranteeing stability of the entire state of the system. We validated this control approach through simulations and experiments by designing tracking controllers for a wheeled mobile robot.

Funding Data

- Office of Naval Research (Grant No. N00014-19-1-2255; Funder ID: 10.13039/100000006).

References

[1] Michel, A. N., Miller, R. K., and Tang, W., 1978, "Lyapunov Stability of Interconnected Systems: Decomposition Into Strongly Connected Subsystems," IEEE Trans. Circuits Syst., **25**, pp. 799–809.

[2] Vidyasagar, M., 1980, "Decomposition Techniques for Large-Scale Systems With Nonadditive Interactions: Stability and Stabilizability," IEEE Trans. Autom. Control, **25**(4), pp. 773–779.

[3] Panteley, E., and Loria, A., 1998, "On Global Uniform Asymptotic Stability of Nonlinear Time-Varying Systems in Cascade," Sys. Control Lett., **33**(2), pp. 131–138.

[4] Sontag, E. D., 1989, "Remarks on Stabilization and Input-to-State Stability," Proceedings of IEEE Conference on Decision and Control, Tampa, FL, Dec. 13–15, pp. 1376–1378.

[5] Sontag, E. D., 1989, "Smooth Stabilization Implies Coprime Factorization," IEEE Trans. Autom. Control, **34**(4), pp. 435–443.

[6] Jiang, Z.-P., and Mareels, I. M. Y., 1997, "A Small-Gain Control Method for Nonlinear Cascaded Systems With Dynamic Uncertainties," IEEE Trans. Autom. Control, **42**(3), pp. 292–308.

[7] Michel, A. N., and Miller, R. K., 1977, *Qualitative Analysis of Large Scale Dynamical Systems*, Academic Press, New York.

[8] Haddad, W. M., and Nersesov, S. G., 2011, *Stability and Control of Large-Scale Dynamical Systems. A Vector Dissipative Systems Approach*, Princeton University Press, Princeton, NJ.

[9] Jiang, Z.-P., Teel, A. R., and Praly, L., 1994, "Small-Gain Theorem for ISS Systems and Applications," Math. Control, Signals, Syst., **7**(2), pp. 95–120.

[10] Šiljak, D. D., 1978, *Large-Scale Dynamic Systems: Stability and Structure*, Elsevier North-Holland, New York.

[11] Nersesov, S. G., and Haddad, W. M., 2006, "On the Stability and Control of Nonlinear Dynamical Systems Via Vector Lyapunov Functions," IEEE Trans. Autom. Control, **51**(2), pp. 203–215.

[12] Vorotnikov, V. I., 1998, *Partial Stability and Control*, Birkhäuser, Boston, MA.

[13] Haddad, W. M., and Chellaboina, V., 2008, *Nonlinear Dynamical Systems and Control. A Lyapunov-Based Approach*, Princeton University Press, Princeton, NJ.

[14] Khalil, H. K., 2015, *Nonlinear Control*, Pearson, Upper Saddle River, NJ.

[15] Nersesov, S. G., and Ashrafiuon, H., 2019, "The Use of Partial Stability in the Analysis of Interconnected Systems," Proceedings of the European Control Conference, Naples, Italy, June 25–28, pp. 155–159.

[16] Khalil, H. K., 2002, *Nonlinear Systems*, 3rd ed., Prentice Hall, Upper Saddle River, NJ.

[17] Apostol, T. M., 1974, *Mathematical Analysis*, Addison-Wesley, Reading, MA.

[18] Ashrafiuon, H., Nersesov, S., and Clayton, G., 2017, "Trajectory Tracking Control of Planar Underactuated Vehicles," Trans. Autom. Control, **62**(4), pp. 1959–1965.

[19] Pomett, J.-B., Thuilot, B., Bastin, G., and Campion, G., 1992, "A Hybrid Strategy for the Feedback Stabilization of Nonholonomic Mobile Robots," Proceedings of the IEEE International Conference on Robotics and Automation, Nice, France, May 12–14, pp. 129–134.