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# Source Seeking for Planar Underactuated Vehicles by Surge Force Tuning

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*Abstract*— **We extend source seeking algorithms, in the absence of position and velocity measurements, and with tuning of the surge input, from velocity-actuated (unicycle) kinematic models to force-actuated generic Euler-Lagrange dynamic underactuated models. In the design and analysis, we employ a symmetric product approximation, averaging, passivity, and partial-state stability theory. The proposed control law requires only real-time measurement of the source signal at the current position of the vehicle and ensures semi-global practical uniform asymptotic stability (SPUAS) with respect to the linear motion coordinates for the closed-loop system. The performance of our source seeker with surge force tuning is illustrated with numerical simulations of an underactuated surface vessel.**

## I. Introduction

Extremum seeking (ES) is a real-time model-free optimization approach that is applicable not only to static maps but also, somewhat uniquely, to dynamical systems [1]. Following the development of the ES convergence guarantees by [2], and their semi-global extension by [3], ES has been a flourishing research area, especially in the domain of autonomous vehicle control for finding sources of signals (electromagnetic, optical, chemical, etc.), distance-based localization, distance-based formation control, etc. In particular, GPS and inertial navigation system (INS) signals are not always available in practice. Hence, autonomous vehicles that operate without GPS or INS benefit from source seeking capabilities.

Most real vehicles are underactuated, where by *underactuated* it is commonly meant that the number of independent actuators of a vehicle is strictly lower than the number of the vehicle's degrees of freedom (DOF), as defined by the dimension of the configuration space [4]. As a consequence of underactuation, the control design for these vehicles is much more difficult than for fully-actuated vehicles [5]. Specifically, fully-actuated *mechanical system* models (comprising the kinematic and dynamic equations) can be feedback linearized into double-integrator dynamics. This is not possible for underactuated vehicles. Furthermore, unlike (first-order) nonholonomic systems, where nonintegrable constraints are imposed on system velocities (such as in the unicycle), underactuated dynamic vehicle models describe the motions constrained by nonintegrable acceleration constraints, and thus, ES algorithms developed for first-order systems cannot be directly applied to underactuated vehicles.

Given the rich variety of model types, spatial dimensions, and input tuning options for autonomous vehicles, numerous approaches for source seeking have emerged in the literature. We categorize the existing results into classical averaging-based [6], [7], Lie bracket averaging-based [8], [9], and symmetric product approximation-based seekers [10], [11] according to different types of averaging techniques. Generally speaking, the classical averaging methods and the Lie bracket approximation approaches cannot be applied directly to a generic second-order (force-controlled) vehicle model—Sect. 3 in [6] illustrates the need for additional compensation and analysis but considers only a fully actuated vehicle.

The *symmetric product approximation* approach, which [12], [13] introduced for vibrational control of mechanical systems, has enabled considerable further advances in force-actuated source seeking. The symmetric product approximation was first employed in source seeking with a force-controlled unicycle in [10] but assuming the availability of velocity measurements. The requirement of velocity measurement was removed by [11]. While these innovative works are the first to employ symmetric product approximation for source seeking, their model of angular motion is simplified. Recently, the symmetric product approximation-based ES algorithm was generalized to a class of affine connection mechanical control systems in [14]. In an alternative pursuit by [15], for fully-actuated dissipation-free vehicles, a symmetric product approximation-based source seeker with a phase-lead compensator injects damping into the system to achieve convergence. In addition, the method applies to systems on Lie groups including two- and three-dimensional vehicle models.

In this paper, we develop a novel source seeking strategy for generic force-controlled planar underactuated vehicles. We prove that the trajectories of a class of underactuated mechanical systems can be approximated by the trajectories of corresponding *symmetric product systems*. Unlike the strategies presented in [6], [15] for fully-actuated vehicles (integrators), the proposed approach can be applied to underactuated vehicles, including marine surface vessels, planar underwater vehicles, etc. The seeking scheme we design does not require any position or velocity measurements. It requires only real-time measurements of the source signal at the current position of the vehicle, and ensures the semi-

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global practical uniform asymptotic stability (SPUAS) with respect to the linear motion coordinates for the closed-loop systems. The structure of the proposed controller is exceptionally simple and easy to implement: the measured output is multiplied by a periodic signal and fed into the surge force.

**Notations.** Let $\mathbb{R}^n$ denote the $n$-dimensional real vector space; $\mathbb{R}_{\geq 0}$ the set of all non-negative real numbers; $|\cdot|$ the Euclidean norm of vectors in $\mathbb{R}^n$. The gradient of a continuously differentiable function $f:\mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla f(x) := \left[\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n}\right]^\top$. For real matrices $A \in \mathbb{R}^{n\times m}$, we use the matrix norm $||A|| = \sup\{|Ax| : |x| = 1\}$. For any constant $r > 0$, we use the notation $\bar{\mathcal{B}}^n_r := \{x \in \mathbb{R}^n : |x| \leq r\}$ to denote a ball of radius $r$ in $\mathbb{R}^n$. For two vector fields $f, g : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, the Lie bracket is denoted by $(\mathrm{ad}_g f)(t,x) = [g, f](t,x) := \frac{\partial f(t,x)}{\partial x} g(t,x) - \frac{\partial g(t,x)}{\partial x} f(t,x)$, and $\mathrm{ad}_g^k f := \mathrm{ad}_g^{k-1}(\mathrm{ad}_g f)$.

## II. Preliminaries and Problem Statement

### A. Model of Planar Underactuated Vehicles

A generic planar underactuated vehicle can be modeled as a 3-DOF planar rigid body with two independent control inputs. Let $\mathcal{F}_s$ denote the fixed inertial frame attached to the ground, and $\mathcal{F}_b$ the body-fixed frame attached to the center of mass of the vehicle. The position of the vehicle in $\mathcal{F}_s$ is described by $(x, y)$, and the orientation of the vehicle is represented by $\theta$, as shown in Fig 1. The equations of motion of the planar underactuated vehicle are given by [5]

$$\dot{q} = J(q)v, \tag{1a}$$

$$M\dot{v} + C(v)v + Dv = Gu, \tag{1b}$$

where $q = [x, y, \theta]^\top \in \mathbb{R}^3$ is the configuration of the vehicle; $v = [v_x, v_y, \omega]^\top \in \mathbb{R}^3$ is the generalized velocity vector consisting of the linear velocity $(v_x, v_y)$ in the body-fixed frame and the angular velocity $\omega$; $u = [u_1, u_2]^\top \in \mathbb{R}^2$ is the control input vector; $J(q)$ is the kinematic transformation matrix given by

$$J(q) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{2}$$

$M = \mathrm{diag}\{m_{11}, m_{22}, m_{33}\} > 0$ is the inertia matrix; $C(v) = -C(v)^\top$ the Coriolis matrix. The components of vector $C(v)v$ are homogeneous polynomials in $(v_x, v_y, \omega)$ of degree 2 [4]. We assume that the damping matrix $D$ is positive definite and constant, which implies that the damping force is proportional to the velocity. We also assume that the surge force and the yaw torque are the two independent control inputs, and accordingly, the input matrix $G$ is given by

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

The system (1a)-(1b) can model a wide class of planar underactuated vehicles such as marine surface vessels, underwater vehicles, etc.

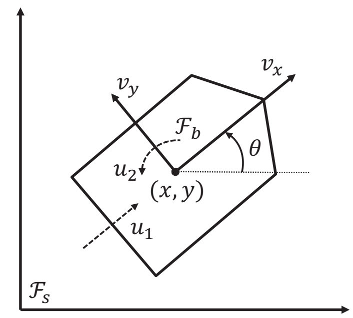


Fig. 1. Top view of the planar underactuated vehicle.

### B. Problem Statement

Assume that the position-dependent nonlinear cost function $\rho : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is smooth and has a global extremum, i.e., there exists a unique $(x^\star, y^\star) \in \mathbb{R}^2$ such that

$$\nabla\rho(x^\star, y^\star) = 0 \text{ and } \nabla\rho(x, y) \neq 0, \forall (x, y) \neq (x^\star, y^\star). \tag{4}$$

In applications, $\rho(\cdot)$ may represent the distance between the vehicle and a source, or the strength of a certain (electromagnetic, optical, etc.) signal. Without loss of generality, we assume that $(x^\star, y^\star)$ is the minimum of the function $\rho(\cdot)$ and the vehicle can measure the value of $\rho(x(t), y(t))$ in real time. Note that both the extremum $(x^\star, y^\star)$ and the gradient $\nabla\rho$ are unknown. Given any constant $\varepsilon > 0$, the objective is to develop a feedback controller to steer the vehicle without position and velocity measurements such that

$$\lim_{t\to\infty} |(x(t), y(t)) - (x^\star, y^\star)| \leq \varepsilon. \tag{5}$$

### C. Shifted Passivity

In the existing literature, there are generally two types of source seeking schemes for vehicle systems: 1) tuning the forward motion of the vehicle by the ES loop while keeping the angular speed constant (e.g., [7], [8], [10]), and 2) tuning the angular motion by the ES loop while keeping the forward velocity constant (e.g., [9]). In either case, the desired (linear/angular) velocity component is not zero, but instead has a steady-state value corresponding to a non-zero constant input. We formulate this property from the viewpoint of *shifted passivity*.

Consider the system (1a)-(1b) with the output $\eta := G^\top v$. Define the steady-state set

$$\mathcal{E} := \{(v, u) \in \mathbb{R}^3 \times \mathbb{R}^2 : C(v)v + Dv - Gu = 0\}. \tag{6}$$

Fix $(v^*, u^*) \in \mathcal{E}$ and the steady-state output $\eta^* := G^\top v^*$.

*Definition 1 (Shifted passivity):* The system (1a)-(1b) is said to be *shifted passive* if the input-output mapping $(u - u^*) \mapsto (\eta - \eta^*)$ is passive, i.e., there exists a storage function $\mathcal{H} : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ such that for all $(v, u) \in \mathbb{R}^3 \times \mathbb{R}^2$,

$$\dot{\mathcal{H}} := (\nabla\mathcal{H}(v))^\top \dot{v} \leq (u - u^*)^\top (\eta - \eta^*). \tag{7}$$

*Proposition 1:* Consider the system (1a)-(1b) with the steady-state input $u^* = [0, c]^\top$, where $c > 0$ is a constant. Then, there exists $\hat{c} > 0$ such that for all $c \in (0, \hat{c})$, the system (1a)-(1b) is shifted passive.

*Proof:* Fix the input $u^* = [0, c]^\top$, and the corresponding steady-state velocity and output are $v^* = [0, 0, \omega^*]^\top$ and $\eta^* = [0, \omega^*]^\top$, respectively. Let the storage function be $\mathcal{H}(v) = \frac{1}{2}(v - v^*)^\top M(v - v^*)$. Then, the time derivative of $\mathcal{H}(v)$ along the trajectories of (1a)-(1b) is given by

$$\begin{aligned}\dot{\mathcal{H}} &= (v - v^*)^\top [G(u - u^*) - C(v)v - Dv + Gu^*] \\ &= (\eta - \eta^*)^\top (u - u^*) - (v - v^*)^\top [C(v)v + Dv - Gu^*] \\ &= (\eta - \eta^*)^\top (u - u^*) - (v - v^*)^\top D(v - v^*) \\ &\quad - (v - v^*)^\top [C(v) - C(v^*)] v^*, \end{aligned} \tag{8}$$

where we used (1b), and added and subtracted the term $Gu^*$ in the first identity, added and subtracted the term $(C(v) + D)v^*$ in the second identity, and used $Gu^* = C(v^*)v^* + Dv^*$ in the third identity. Let us denote $\mathcal{J}(v) := C(v)v^* + Dv$. Then, from (8) we have

$$\dot{\mathcal{H}} = (\eta - \eta^*)^\top (u - u^*) - (v - v^*)^\top [\mathcal{J}(v) - \mathcal{J}(v^*)]. \tag{9}$$

It follows from the homogeneity of $C(v)v$ that for all $v \in \mathbb{R}^3$, $||\partial [C(v)e_3] / \partial v||$ is bounded, where $e_3 = [0, 0, 1]^\top$. Thus, we can always choose $\omega^*$ small enough such that $\partial [C(v)v^*] / \partial v + [\partial [C(v)v^*] / \partial v]^\top \leq 2D$, which implies that $(\partial \mathcal{J}(v) / \partial v) + (\partial \mathcal{J}(v) / \partial v)^\top \geq 0$ for all $v \in \mathbb{R}^3$. Therefore, the map $\mathcal{J}(\cdot)$ is monotone, and correspondingly, $(v - v^*)^\top [\mathcal{J}(v) - \mathcal{J}(v^*)] \geq 0$, which completes the proof. ■

### D. Partial-State Practical Stability

Consider the nonlinear interconnected system

$$\dot{x}_1 = f_1(x_1, x_2), \quad x_1(t_0) = x_{10}, \quad t \geq t_0, \tag{10}$$
$$\dot{x}_2 = f_2(x_1, x_2), \quad x_2(t_0) = x_{20}, \tag{11}$$

where $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(x_1, x_2)$ is locally Lipschitz in $x_1$ uniformly in $x_2$; $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that for every $x_1 \in \mathbb{R}^{n_1}$, $f_2(x_1, x_2)$ is locally Lipschitz in $x_2$ uniformly in $x_1$. Let $x_1(\cdot) := x_1(\cdot, x_{10}, x_{20})$ and $x_2(\cdot) := x_2(\cdot, x_{10}, x_{20})$ denote the solution of the initial value problem (10)-(11).We define the *partial-state stability* as stability with respect to $x_1$ for the interconnected system (10)-(11), which is also referred to as "partial stability" in the literature [16].

*Definition 2 (P-UGAS):* The system (10)-(11) is *globally asymptotically stable (GAS) with respect to $x_1$ uniformly in $x_{20}$* if the following conditions are satisfied:

1) *Partial-State Uniform Stability (P-US):* For each $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that
$$|x_{10}| \leq \delta(\varepsilon) \implies |x_1(t)| \leq \varepsilon, \quad \forall t \geq 0, \ \forall x_{20} \in \mathbb{R}^{n_2}.$$
2) *Partial-State Uniform Global Boundedness (P-UGB):* For each $r > 0$, there exists $R(r)$ such that
$$|x_{10}| \leq r \implies |x_1(t)| \leq R(r), \quad \forall t \geq 0, \ \forall x_{20} \in \mathbb{R}^{n_2}.$$
3) *Partial-State Uniform Global Attractivity (P-UGA):* For each $r > 0$, for each $\sigma > 0$, there exists $T(r, \sigma)$ such that
$$|x_{10}| \leq r \implies |x_1(t)| \leq \sigma, \quad \forall t \geq T(r, \sigma), \ \forall x_{20} \in \mathbb{R}^{n_2}.$$

We present Lyapunov conditions for P-UG(A)S of (10)-(11). Given a function $V(x_1, x_2)$, define $\dot{V}(x_1, x_2) := (\partial V / \partial x) f(x_1, x_2)$, where $x := [x_1^\top, x_2^\top]^\top$ and $f(x_1, x_2) := [f_1(x_1, x_2)^\top, f_2(x_1, x_2)^\top]^\top$.

*Theorem 1 ( [17]):* Consider the interconnected system (10)-(11). If there exist functions $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ of class $C^1$, class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2$ such that for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\alpha_1(|x_1|) \leq V(x_1, x_2) \leq \alpha_2(|x_1|), \tag{12}$$
$$\dot{V}(x_1, x_2) \leq 0, \tag{13}$$

then the system (10)-(11) is US and UGB with respect to $x_1$ uniformly in $x_{20}$. Furthermore, if exists a positive definite function $\alpha_3$ such that

$$\dot{V}(x_1, x_2) \leq -\alpha_3(|x_1|), \tag{14}$$

then the system (10)-(11) is UGAS with respect to $x_1$ uniformly in $x_{20}$.

Next, we define *partial-state practical stability* for interconnected systems that depends on a small parameter $\varepsilon > 0$,

$$\dot{x}_1 = f_1^\varepsilon(t, x_1, x_2), \quad x_1^\varepsilon(t_0) = x_{10}, \quad t \geq t_0, \tag{15}$$
$$\dot{x}_2 = f_2^\varepsilon(t, x_1, x_2), \quad x_2^\varepsilon(t_0) = x_{20}. \tag{16}$$

Let $x_1^\varepsilon(\cdot) := x_1^\varepsilon(\cdot, t_0, x_{10}, x_{20})$ and $x_2^\varepsilon(\cdot) := x_2^\varepsilon(\cdot, t_0, x_{10}, x_{20})$ denote the solution of the initial value problem (15)-(16).

*Definition 3 (P-SPUAS):* The system (15)-(16) said to be *semi-globally practically asymptotically stable (SPAS) with respect to $x_1$ uniformly in* $(t_0, x_{20})$ if for every compact set $\bar{\mathcal{B}}_r^{n_2} \subset \mathbb{R}^{n_2}$, the following conditions are satisfied:

1) For every $c_2 > 0$, there exists $c_1$ and $\hat{\varepsilon}(r) > 0$ such that for all $(t_0, x_{20}) \in \mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r^{n_2}$ and for all $\varepsilon \in (0, \hat{\varepsilon})$,
$$|x_{10}| \leq c_1 \implies |x_1^\varepsilon(t)| \leq c_2, \quad \forall t \geq t_0.$$
2) For every $c_1 > 0$, there exists $c_2$ and $\hat{\varepsilon}(r) > 0$ such that for all $(t_0, x_{20}) \in \mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r^{n_2}$ and for all $\varepsilon \in (0, \hat{\varepsilon})$,
$$|x_{10}| \leq c_1 \implies |x_1^\varepsilon(t)| \leq c_2, \quad \forall t \geq t_0.$$
3) For all $c_1 > 0$, $c_2 > 0$, there exists $T(c_1, c_2)$ and $\hat{\varepsilon}(r) > 0$ such that for all $(t_0, x_{20}) \in \mathbb{R}_{\geq 0} \times \bar{\mathcal{B}}_r^{n_2}$ and for all $\varepsilon \in (0, \hat{\varepsilon})$,
$$|x_{10}| \leq c_1 \implies |x_1^\varepsilon(t)| \leq c_2, \quad \forall t \geq t_0 + T(c_1, c_2).$$

*Definition 4 (Partial Converging Trajectories Property):* The systems (10)-(11) and (15)-(16) are said to satisfy the *partial converging trajectories property* if for every $T > 0$, for every compact set $K \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and for every $d > 0$, there exists $\varepsilon^*$ such that for all $t_0 \geq 0$, for all $(x_{10}, x_{20}) \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$,

$$|x_1^\varepsilon(t) - x_1(t)| < d, \quad \forall t \in [t_0, t_0 + T]. \tag{17}$$

*Proposition 2:* Assume that for the system (10)-(11), the flow $(x_1(\cdot), x_2(\cdot))$ is forward complete, and that the systems (10)-(11) and (15)-(16) satisfy the partial converging trajectories property. If (10)-(11) is GAS with respect to $x_1$ uniformly in $x_{20}$, then (15)-(16) is SPAS with respect to $x_1$ uniformly in $(t_0, x_{20})$.

## III. SYMMETRIC PRODUCT APPROXIMATIONS

Consider the system (1a)-(1b). Let the input vector be

$$u = b_0 + \frac{1}{\varepsilon}\sum_{i=1}^{m} b_i(q) w_i\left(\frac{t}{\varepsilon}\right), \tag{18}$$

where $\varepsilon$ is a positive constant, $m$ is a positive integer, $b_0 = [b_{10}, b_{20}]^\top$ is a constant vector, $b_i(q) = [b_{1i}(q), b_{2i}(q)]^\top$, and $\{w_i(t)\}$ are $T$-periodic functions satisfying

$$\int_0^T w_i(s_1)\mathrm{d}s_1 = 0, \quad i = 1,\ldots,m, \tag{19}$$

$$\int_0^T \int_0^{s_2} w_i(s_1)\mathrm{d}s_1\mathrm{d}s_2 = 0, \quad i = 1,\ldots,m. \tag{20}$$

Then, (1a)-(1b) with the input vector (18) in time scale $\tau = t/\varepsilon$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} q \\ v \end{bmatrix} = \varepsilon \underbrace{\begin{bmatrix} J(q)v \\ -M^{-1}[C(v)v + Dv - B_0] \end{bmatrix}}_{\mathsf{f}(q,v)} + \underbrace{\begin{bmatrix} 0 \\ \sum_{i=1}^{m} B_i(q) w_i(\tau) \end{bmatrix}}_{\mathsf{g}(\tau,q)}, \tag{21}$$

where $B_0 = Gb_0$ and $B_i(q) = M^{-1}Gb_i(q)$ for $i = 1,\ldots,m$. Denote $\mathsf{f}_2(v) = -M^{-1}[C(v)v + Dv - B_0]$. The *symmetric product* of two vector fields $X, Y : \mathbb{R}^3 \to \mathbb{R}^3$ corresponding to the system (1a)-(1b) is defined as

$$\langle X : Y \rangle = \frac{\partial X}{\partial q} J(q) Y(q) + \frac{\partial Y}{\partial q} J(q) X(q) - \left(\frac{\partial}{\partial v}\left(\frac{\partial \mathsf{f}_2}{\partial v} X(q)\right)\right) Y(q). \tag{22}$$

In the next theorem, we show that the trajectories of system (1a)-(1b) with the input vector (18) can be approximated by the trajectories of the *symmetric product system*

$$\dot{\bar{q}} = J(\bar{q})\bar{v}, \tag{23a}$$

$$M\dot{\bar{v}} + C(\bar{v})\bar{v} + D\bar{v} = B_0 - M \sum_{i,j=1}^{m} \Lambda_{ij} \langle B_i : B_j \rangle(\bar{q}), \tag{23b}$$

where

$$\Lambda_{ij} = \frac{1}{2T}\int_0^T \left(\int_0^{s_1} w_i(s_2)\mathrm{d}s_2\right)\left(\int_0^{s_1} w_j(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1. \tag{24}$$

Define the time-varying vector field as

$$\Xi(t,q) := \sum_{i=1}^{m}\left(\int_0^t w_i(s)\mathrm{d}s\right) B_i(q). \tag{25}$$

*Theorem 2:* Consider the system (1a)-(1b) with input vector (18) and the symmetric product system (23a)-(23b). Assume that the initial conditions of the two systems are the same. Denote the solutions of (1a)-(1b) and (23a)-(23b) as $(q(t), v(t))$ and $(\bar{q}(t), \bar{v}(t))$ for $t \geq 0$, respectively. If the system (23a)-(23b) is GAS with respect to $(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y)$ uniformly in $(\bar{\theta}(0), \bar{\omega}(0))$, then the system (1a)-(1b) is semi-globally practically asymptotically stable (SPAS) with respect to $(x, y, v_x, v_y)$ uniformly in $(\theta(0), \omega(0))$.

*Proof:* By the variation of constants formula in Appendix A, the pull back system of (21) is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \hat{q} \\ \hat{v} \end{bmatrix} = \varepsilon \mathsf{F}(\tau, \hat{q}, \hat{v}), \tag{26}$$

where $(\hat{q}(0), \hat{v}(0)) = (q(0), v(0))$ and

$$\mathsf{F}(\tau,q,v) = \mathsf{f}(q,v) + \sum_{k=1}^{\infty}\int_0^{\tau}\cdots\int_0^{s_{k-1}}\left(\mathrm{ad}_{\mathsf{g}(s_k,q)}\cdots\mathrm{ad}_{\mathsf{g}(s_1,q)}\mathsf{f}(q,v)\right)\mathrm{d}s_k\cdots\mathrm{d}s_1.$$

By direct calculations, we have

$$\mathrm{ad}_{\mathsf{g}(s_1,q)}\mathsf{f}(q,v) = \sum_{i=1}^{m} w_i(s_1)\begin{bmatrix} J(q)B_i(q) \\ \left(\frac{\partial \mathsf{f}_2(v)}{\partial v}\right) B_i(q) - \left(\frac{\partial B_i}{\partial q}\right) J(q)v \end{bmatrix},$$

and

$$\mathrm{ad}_{\mathsf{g}(s_2,q)}\mathrm{ad}_{\mathsf{g}(s_1,q)}\mathsf{f}(q,v) = -\sum_{i,j=1}^{m} w_i(s_1)w_j(s_2)\begin{bmatrix} 0 \\ \langle B_i : B_j\rangle(q) \end{bmatrix}.$$

Note that the symmetric product $\langle B_i : B_j \rangle(q)$ is a vector field depending only on $q$. Thus, the higher order terms $\mathrm{ad}_{\mathsf{g}(s_k,q)}\cdots\mathrm{ad}_{\mathsf{g}(s_1,q)}\mathsf{f}(q,v) \equiv 0$ for all $k \geq 3$. The pull back vector field $\mathsf{F}(\tau,q,v)$ is given by

$$\begin{aligned}\mathsf{F} =& \mathsf{f}(q,v) + \sum_{i=1}^{m}\begin{bmatrix} J(q)B_i(q) \\ \left(\frac{\partial \mathsf{f}_2}{\partial v}\right) B_i - \left(\frac{\partial B_i}{\partial q}\right) J(q)v \end{bmatrix}\int_0^{\tau} w_i(s_1)\mathrm{d}s_1 \\ &- \sum_{i,j=1}^{m}\begin{bmatrix} 0 \\ \langle B_i : B_j \rangle \end{bmatrix}\int_0^{\tau}\int_0^{s_1} w_i(s_1)w_j(s_2)\mathrm{d}s_2\mathrm{d}s_1. \end{aligned}\tag{27}$$

Denote the solution of the pull back system (26) by $(\hat{q}(\tau), \hat{v}(\tau))$. Then, it follows from Theorem 4 that the solution to (21) is given by the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \sum_{i=1}^{m} B_i(q) w_i(\tau) \end{bmatrix}, \quad \begin{bmatrix} q(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} \hat{q}(\tau) \\ \hat{v}(\tau) \end{bmatrix}. \tag{28}$$

Therefore, we have

$$q(\tau) \equiv q(0) \equiv \hat{q}(\tau), \tag{29}$$

$$v(\tau) = \hat{v}(\tau) + \Xi(\tau, q(\tau)). \tag{30}$$

The pull back system (26) is in the standard averaging form. Consider the averaged system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \bar{q} \\ \bar{v} \end{bmatrix} = \frac{\varepsilon}{T}\int_0^T \mathsf{F}(\tau, \bar{q}, \bar{v})\mathrm{d}\tau, \tag{31}$$

and denote the solution by $(\bar{q}(t), \bar{v}(t))$. It follows from (20), the symmetry of symmetric product, and integration by parts, that the averaged system (31) in time scale $t = \varepsilon\tau$ is the symmetric product system (23a)-(23b).

According to the averaging theorem [18, Theorem 10.4], there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$,

$$|\hat{q}(t) - \bar{q}(t)| = O(\varepsilon), \quad \text{and} \quad |\hat{v}(t) - \bar{v}(t)| = O(\varepsilon) \tag{32}$$

as $\varepsilon \to 0$ on time scale 1. We recover the partial converging trajectories property by substituting (29)-(30) into (32). Finally, it follows directly from Proposition 2 that the system

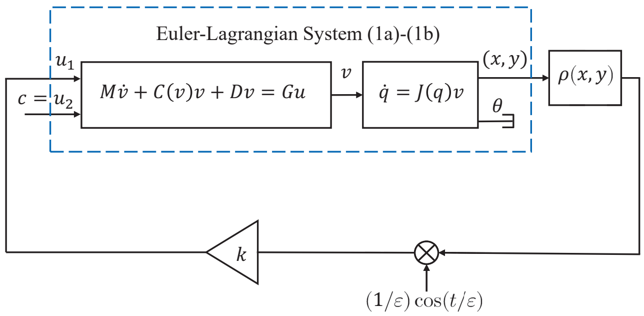


Fig. 2. Source seeking scheme for planar vehicle system (1a)-(1b).

(1a)-(1b) is SPAS with respect to $(x, y, v_x, v_y)$ uniformly in $(\theta(0), \omega(0))$, which completes the proof. ■

*Remark 1:* In Theorem 2, instead of requiring UGAS of the symmetric product system as in standard averaging theory [19], we only assume (23a)-(23b) to be P-UGAS with respect to $(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y)$, while the remaining part of the state $(\bar{\theta}(t), \bar{\omega}(t))$ does not necessarily converge to $(0,0)$. Correspondingly, in the source seeking design in the next section, the approximation in the linear motion $|(x, y, v_x, v_y) - (x^\star, y^\star, 0, 0)| = O(\varepsilon)$ is valid for all $t \geq 0$, while the angular motion of the vehicle can be persistently exciting.

## IV. Source Seeking for Underactuated Vehicles

### A. Source Seeking Scheme

We propose a source seeking scheme for the planar vehicle system (1a)-(1b) as it is depicted in Fig. 2. In the proposed scheme, the surge force of the vehicle is tuned by the ES loop, while the yaw torque keeps a certain constant value. The proposed surge force tuning based source seeking scheme is similar to the methods in [7], [8], [11], but will be analyzed in the symmetric product approximation framework.

The control law in Fig. 2 is given by

$$u_1 = \frac{k}{\varepsilon} \cos\left(\frac{t}{\varepsilon}\right) \rho(x, y), \tag{33}$$

$$u_2 = c, \tag{34}$$

where $\varepsilon$, $k$, and $c$ are positive parameters. The gain $k$ is used to tune the transient performance. The small parameter $\varepsilon$ introduces the "high-magnitude high-frequency force", which leads to the symmetric product approximation. The constant torque $c$ maintains a persistently exciting angular motion of the vehicle, which is necessary to establish convergence for underactuated vehicle systems.

### B. Stability Analysis

*Theorem 3:* Consider the system (1a)-(1b) with inputs (33)-(34). Suppose that the cost function $\rho(x, y) \geq 0$ satisfies (4). Then, there exists $\hat{c} > 0$ and for any $c \in (0, \hat{c})$, there exists $\hat{\varepsilon} > 0$ such that for the given $c$ and any $\varepsilon \in (0, \hat{\varepsilon})$ and $k > 0$, the closed-loop system is SPAS with respect to $(x - x^\star, y - y^\star, v_x, v_y)$ uniformly in $(\theta(0), \omega(0))$.

*Proof:* Note that the control law (33)-(34) is in the form of (18), where $m = 1$, $b_0 = [0, c]^\top$, $b_1(q) = [k\rho(x, y), 0]^\top$, and $w_1(t) = \cos(t)$. It can be verified that conditions (19)-(20) hold for $T = 2\pi$. Thus, it follows from Theorem 2 that the closed-loop system is SPAS with respect to $(x - x^\star, y - y^\star, v_x, v_y)$ uniformly in $(\theta(0), \omega(0))$ if the corresponding symmetric product system (23a)-(23b) is GAS with respect to $(\bar{x} - x^\star, \bar{y} - y^\star, \bar{v}_x, \bar{v}_y)$ uniformly in $(\theta(0), \omega(0))$. Next, we show that it is indeed the case.

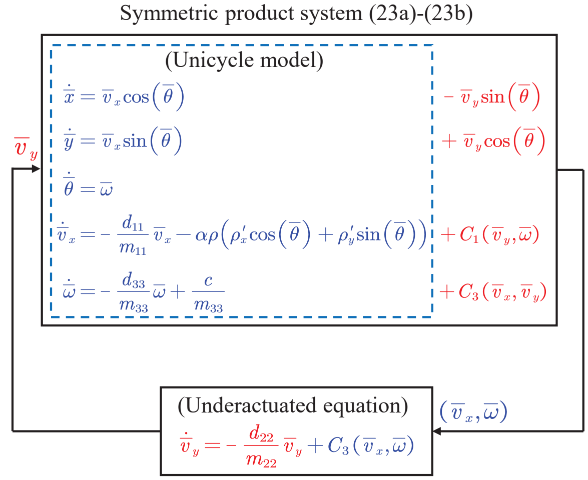


Fig. 3. Feedback interconnection of the symmetric product system (23a)-(23b).

By direct calculations, we have $\Lambda_{11} = 1/4$, and the symmetric product is given by $\langle B_1 : B_1 \rangle(\bar{q}) = 2(m_{11}^{-1}k)^2 \rho(\bar{x}, \bar{y})[\rho_x'(\bar{x}, \bar{y})\cos(\bar{\theta}) + \rho_y'(\bar{x}, \bar{y})\sin(\bar{\theta}), 0, 0]^\top$, where $\rho_x'(x, y) := \partial\rho(x, y)/\partial x$ and $\rho_y'(x, y) := \partial\rho(x, y)/\partial y$. The constant torque $c$ is selected such that Proposition 1 holds, and then, the system (23a)-(23b) is shifted passive under the steady-state input $u^* = b_0$. Denote $\alpha = (m_{11}^{-1}k)^2/2$ and $C_i(\cdot)$ the $i$-th component of the vector $-M^{-1}C(\bar{v})\bar{v}$. The symmetric product system can be viewed as a feedback interconnection of two subsystems, as shown in Fig. 3.

When the input $\bar{v}_y \equiv 0$, the nominal system of the upper subsystem is exact the unicycle model under a passive feedback. We first prove the nominal system of the upper subsystem (i.e., $\bar{v}_y \equiv 0$) is P-UGAS. Let $V_1(\bar{x}, \bar{y}, \bar{v}_x) = \frac{1}{2}\bar{v}_x^2 + \frac{\alpha}{2}\rho(\bar{x}, \bar{y})^2$, and along trajectories of the nominal system, we have $\dot{V}_1\big|_{\mathtt{nominal}} = -\frac{d_{11}}{m_{11}}\bar{v}_x^2 \leq 0$, which, according to Theorem 1, shows that the nominal system is US and UGB with respect to $(\bar{x} - x^\star, \bar{y} - y^\star, \bar{v}_x)$ uniformly in $(\bar{\theta}(0), \bar{\omega}(0))$. Then, consider the auxiliary function $V_2 = \bar{v}_x \rho(\bar{x}, \bar{y})(\rho_x' \cos(\bar{\theta}) + \rho_y' \sin(\bar{\theta}))$. Evaluating the time derivative of $V_2$ along trajectories of the nominal system on the set $\{\bar{v}_x = 0\}$, we have $\dot{V}_2\big|_{\mathtt{nominal}, \bar{v}_x = 0} = -\alpha\rho^2(\rho_x' \cos(\bar{\theta}) + \rho_y' \sin(\bar{\theta}))^2$, which is non-zero definite. It follows from Matrosov' theorem [5], [16] that the nominal system is UGAS with respect to $(\bar{x} - x^\star, \bar{y} - y^\star, \bar{v}_x)$ uniformly in $(\bar{\theta}(0), \bar{\omega}(0))$.

Second, we prove that the upper subsystem is input-to-output stable (IOS) by viewing $\bar{v}_y$ as input and $(\bar{v}_x, \bar{\omega})$ as output. Because the nominal part of the upper subsystem is P-UGAS, for each $r > 0$, there exists a constant $\delta_r > 0$ such that for all initial conditions starting in the ball centering at the equilibrium with radius $r$, we have $\max\{|\rho'_x\cos(\bar{\theta})+\rho'_y\sin(\bar{\theta})|, |\rho'_x\cos(\bar{\theta})+\rho'_y\sin(\bar{\theta})|^2, |c\rho(\rho'_x\sin(\bar{\theta})-\rho'_y\cos(\bar{\theta}))|/d_{33}, |\rho(\rho''_{xx}\cos(\bar{\theta})^2+2\rho''_{xy}\sin(\bar{\theta}))\cos(\bar{\theta})|\} < \delta_r$. Let $\mathcal{V}_r = \beta_r V_1 + V_2$, where $\beta_r > 0$ is a constant to be determined. It follows from Young's inequality $ab \le a^2/(2\epsilon)+(\epsilon b^2)/2$ that $\mathcal{V}_r > 0$ and $\dot{\mathcal{V}}_r\big|_{\texttt{nominal}} \le -\bar{v}_x^2 - \frac{\alpha}{2}\rho^2(\rho'_x\cos(\bar{\theta})+\rho'_y\sin(\bar{\theta}))^2+\bar{v}_x\delta_r$ by selecting $\beta_r > \max\left\{\delta_r^2/\alpha, 1+2m_{11}\delta_r/d_{11}+d_{11}/(2\alpha m_{11})\right\}$. Then, taking time derivative of $\mathcal{V}_r$ along trajectories of the upper subsystem, and noting that the quadratic terms $-\bar{v}_x^2-\frac{\alpha}{2}\rho^2(\rho'_x\cos(\bar{\theta})+\rho'_y\sin(\bar{\theta}))^2$ dominate $\dot{\mathcal{V}}_r\big|_{\texttt{upper}}$ when $|(\bar{v}_x, \rho(\rho'_x\cos(\bar{\theta})+\rho'_y\sin(\bar{\theta})))|$ are large, we conclude that the upper subsystem is IOS with input $\bar{v}_y$ and output $(\bar{v}_x, \bar{\omega})$.

Due to the fact that the lower subsystem in Fig. 3 is a stable linear system, it is also IOS by viewing $(\bar{v}_x, \bar{\omega})$ as the input and $\bar{v}_y$ as the output, and the IOS-gain can be rendered arbitrarily small by selecting $c$ small enough. Therefore, the symmetric product system (23a)-(23b) is a feedback interconnection of two IOS subsystems, where the zero-state detectablility can be easily verified. It follows from the small-gain theorem [20] that, there exists $\hat{c} > 0$ such that the symmetric product system (23a)-(23b) is GAS with respect to $(\bar{x}-x^\star, \bar{y}-y^\star, \bar{v}_x, \bar{v}_y)$ uniformly in $(\theta(0), \omega(0))$ for all $c \in (0, \hat{c})$. Finally, we conclude that the closed-loop system is SPAS with respect to $(x-x^\star, y-y^\star, v_x, v_y)$ uniformly in $(\theta(0), \omega(0))$ by invoking Theorem 2. ■

*Remark 2:* Compared with the surge force tuning based source seeking schemes in [7], [8], the presented scheme does not require an additive periodic perturbation. The additive periodic perturbation is necessary in the Lie bracket averaging-based algorithm [8] since it is used to introduce the back-and-forth motion of a vehicle. However, as shown in Section III, only with a multiplicative periodic perturbation and without an additive periodic perturbation, the pull back system still involves an operation that is calculating Lie bracket with the vector $\mathsf{g}$, i.e., $[\mathsf{g}(s_2,q), [\mathsf{g}(s_1,q), \mathsf{f}(q,v)]]$. This viewpoint shows that besides the standard averaging approach [2] and the Lie bracket approximation approach [8], the symmetric product approximation can also be used to obtain gradient information.

## V. Numerical Simulations

In this section, we present simulation results to validate the effectiveness and illustrate the performance of the proposed source seeking scheme. All the parameters are given in SI units.

Consider an underactuated marine surface vessel with linear hydrodynamic damping [5], where the dynamical equations are given by (1a)-(1b) with

$$C(v) = \begin{bmatrix} 0 & 0 & -m_{22}v_y \\ 0 & 0 & m_{11}v_x \\ m_{22}v_y & -m_{11}v_x & 0 \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix},$$

where

$$m_{11} = 1.412, \quad m_{22} = 1.982, \quad m_{33} = 0.354,$$
$$d_{11} = 3.436, \quad d_{22} = 12.99, \quad d_{33} = 0.864.$$

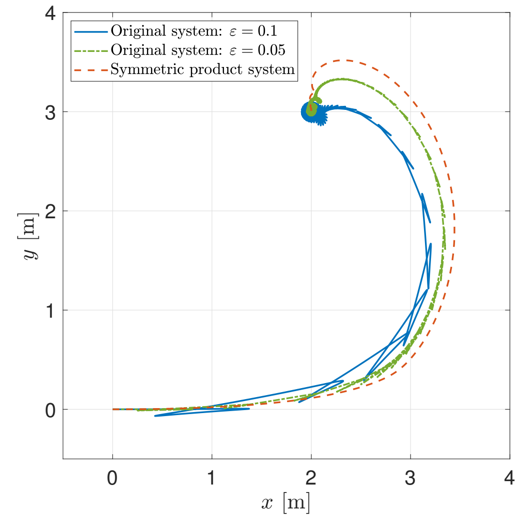


Fig. 4. Paths of the underactuated surface vessel in source seeking ($c = 1$).

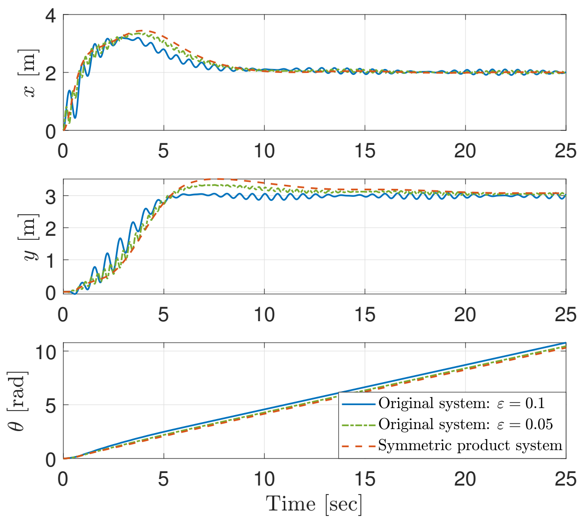


Fig. 5. Configuration trajectories of the underactuated surface vessel in source seeking ($c = 1$).

The surface vessel is assumed to rest at the origin initially, i.e., $(q(0), v(0)) = (0, 0)$. Assume that the cost function is $\rho(x,y) = (x-2)^2 + 0.5(y-3)^2 + 1$.

It follows from the equations (1a)-(1b) that the constant input $(u_1^*, u_2^*) = (0, c)$ leads to the steady-state velocity $v^* = (v_x^*, v_y^*, \omega^*) = (0, 0, c/d_{33})$. Then, the constant $c$ is chosen such that $\partial\left[C(v)v^*\right]/\partial v + \left[\partial\left[C(v)v^*\right]/\partial v\right]^\top \le 2D$ holds. By direct calculation, we have $4d_{11}d_{22} - (\omega^*)^2(m_{11}-m_{22})^2 \ge 0$, which implies that $c \le 2\sqrt{d_{11}d_{22}}d_{33}/(m_{22}-m_{11}) = 20.25$. That is, with the steady-state input $u^* = [0, c]^\top$ with $c \le 20.25$, the system is shifted passive.

In the first example, we select the control parameters in (33)-(34) to be $c = 1$, $k = 1$. The simulation results are shown in Figs. 4-5 for $\varepsilon = 0.1$ and $\varepsilon = 0.05$. In the second example, we increase the constant torque to $c = 3$. The simulation

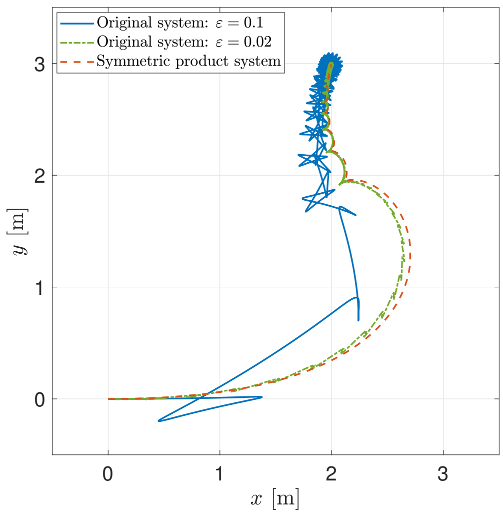


Fig. 6. Paths of the underactuated surface vessel in source seeking $(c = 3)$.

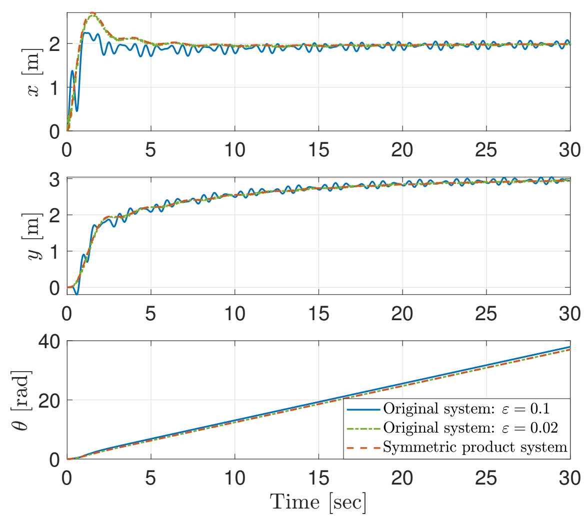


Fig. 7. Configuration trajectories of the underactuated surface vessel in source seeking $(c = 3)$.

results of the second example are shown in Figs. 6-7 for $\varepsilon = 0.1$ and $\varepsilon = 0.02$. It can be seen from both examples that the position trajectory of the underactuated surface vessel converges to the $O(\varepsilon)$-neighborhood of the desired position $(x^\star, y^\star) = (2,3)$. Furthermore, as $\varepsilon \to 0$, the trajectories of the surface vessel converge to the trajectory of symmetric product system which represents the ideal solution. In general, a smaller $\varepsilon$ leads to a smoother trajectory. The only limitation on the value of $\varepsilon$ is the value of the control input (33) which increases as $\varepsilon$ decreases. In both examples, the vessel converges to the desired neighborhood within 30 seconds.

## VI. Conclusions

The ES design for force-controlled underactuated mechanical systems without position or velocity measurements was previously an open problem. In this work, we developed a source seeking scheme for generic force-controlled planar underactuated vehicles by surge force tuning. The control design is based on the symmetric product approximations, averaging, passivity, and partial-state stability theory. The controller does not require any position or velocity measurements but only real-time measurements of the source signal at the current position. The partial-state semi-global practical uniform asymptotic stability (P-SPUAS) is proved for the closed-loop source seeking system. Numerical simulations of an underactuated surface vessel illustrate the performance of the proposed source seeker.

## Appendix A. The Variation of Constants Formula

Consider the dynamical system

$$\dot{x} = g(t,x), \quad x(0) = x_0, \tag{35}$$

where the vector field $g(t,x)$ is locally Lipschitz in $x$ uniformly in $t$. The *flow map* $\Phi^g_{0,t}(\cdot)$ is a diffeomorphism, which describes the solution of (35) at time $t$, i.e., $x(t) = \Phi^g_{0,t}(x_0)$.

Given a diffeomorphism $\phi$ and a vector field $f$, the *pull back* of $f$ along $\phi$, denoted by $\phi^* f$, is the vector field

$$(\phi^* f)(x) := \left(\frac{\partial \phi^{-1}}{\partial x} \circ f \circ \phi\right)(x), \tag{36}$$

where $(f \circ \phi)(x) = f(\phi(x))$. The variation of constants formula [4], [13] characterizes the relationship between the flow of $f + g$ and the flows of $f$ and $g$.

*Theorem 4 (Variation of constants formula):* Consider the dynamical system

$$\dot{x} = f(t,x) + g(t,x), \quad x(0) = x_0, \tag{37}$$

where $f, g : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. If $z(t)$ is the solution of the system

$$\dot{z}(t) = \left(\left(\Phi^g_{0,t}\right)^* f\right)(t,z), \quad z(0) = x_0, \tag{38}$$

then the solution $x(t)$ of the initial value problem

$$\dot{x} = g(t,x), \quad x(0) = z(t) \tag{39}$$

is the solution of system (37).

System (38) is called the *pull back system*. Furthermore, if $f$ is a time-invariant vector field and $g$ is a time-varying vector field, then the pull back of $f$ along $\Phi^g_{0,t}$ is given by

$$\begin{aligned} \left(\left(\Phi^g_{0,t}\right)^* f\right)(t,x) &= f(x) \\ &+ \sum_{k=1}^{\infty} \int_0^t \cdots \int_0^{s_{k-1}} \left(\mathrm{ad}_{g(s_k,x)} \cdots \mathrm{ad}_{g(s_1,x)} f(x)\right) \mathrm{d}s_k \cdots \mathrm{d}s_1 . \end{aligned} \tag{40}$$

## Appendix B.
## Proof of Proposition 2

We successively prove that conditions 1, 2, and 3 of Definition 3 are satisfied.

1) Take an arbitrary $c_2 > 0$, and let $b_2 \in (0, c_2)$. By the P-US property, there exists $c_1$ such that

$$|x_{10}| \le c_1 \implies |x_1(t)| \le b_2, \quad \forall t \ge t_0,\ \forall x_{20} \in \mathbb{R}^{n_2}.$$

Let $b_1 \in (0, c_1)$, and by the P-UGA property, there exists $T$ such that

$$|x_{10}| \le c_1 \implies |x_1(t)| \le b_1, \quad \forall t \ge t_0 + T,\ \forall x_{20} \in \mathbb{R}^{n_2}.$$

Let $d = \min\{c_1 - b_1, c_2 - b_2\}$ and $K = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : |x_1| \le c_1, |x_2| \le r\}$, where $r > 0$ is an arbitrary number. By the partial converging trajectory property, there exists $\varepsilon^*$ such that for all $(x_{10}, x_{20}) \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$,

$$|x_1^\varepsilon(t) - x_1(t)| < d, \quad \forall t \in [t_0, t_0 + T].$$

Thus, we conclude that for all $t_0 \in \mathbb{R}_{\ge 0}$, for all $(x_{10}, x_{20}) \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$,

$$\begin{aligned} &|x_1^\varepsilon(t)| < c_2, \quad \forall t \in [t_0, t_0 + T], \\ &|x_1^\varepsilon(t)| < c_1, \quad \text{for } t = t_0 + T. \end{aligned} \tag{41}$$

Since $|x_1^\varepsilon(t_0 + T)| < c_1$, a repeated application of (41) yields that for all $(x_{10}, x_{20}) \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$, we have $|x_1^\varepsilon(t)| < c_2$, $\forall t \ge t_0$.

2) Take an arbitrary $c_1 > 0$, and let $b_1 \in (0, c_1)$. By the P-UGB and P-UGA properties, there exist $b_2$ and $T$ such that for all $t_0 \in \mathbb{R}_{\ge 0}$ and for all $x_{20} \in \mathbb{R}^{n_2}$,

$$\begin{aligned} &|x_{10}| \le c_1 \implies |x_1(t)| \le b_2, \quad \forall t \ge t_0, \\ &|x_{10}| \le c_1 \implies |x_1(t)| \le b_1, \quad \forall t \ge t_0 + T. \end{aligned}$$

Let $c_2 > b_2$, and by the partial converging trajectory property again, we conclude that there exists $\varepsilon^*$ such that for all $(x_{10}, x_{20}) \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$, we have $|x_1^\varepsilon(t)| < c_2$, $\forall t \ge t_0$.

3) Take arbitrary $c_1, c_2 > 0$. By the Item 1 proven above, there exist $c_3$ and $\varepsilon^*$ such that for all $t_0 \in \mathbb{R}_{\ge 0}$, for all $\varepsilon \in (0, \varepsilon^*)$,

$$|x_{10}| \le c_3 \implies |x_1^\varepsilon(t)| < c_2, \quad \forall t \ge t_0,\ \forall x_{20} \in \bar{\mathcal{B}}_r^{n_2}. \tag{42}$$

Let $b_3 \in (0, c_3)$, and by the P-UGA property, there exists $T$ such that for all $x_{20} \in \mathbb{R}^{n_2}$,

$$|x_{10}| \le c_1 \implies |x_1(t)| \le b_3, \quad \forall t \ge t_0 + T.$$

Let $d = c_3 - b_3$. Then, by the partial converging trajectory property, there exists $\varepsilon^\#$ such that for all $\varepsilon \in (0, \varepsilon^\#)$ and for all $x_{20} \in \bar{\mathcal{B}}_r^{n_2}$,

$$|x_{10}| \le c_1 \implies |x_1^\varepsilon(t) - x_1(t)| < d, \quad \forall t \in [t_0, t_0 + T],$$

which implies that for all $\varepsilon \in (0, \varepsilon^\#)$ and for all $x_{20} \in \bar{\mathcal{B}}_r^{n_2}$,

$$|x_{10}| \le c_1 \implies |x_1^\varepsilon(t_0 + T)| < c_3.$$

Finally, together with (42), we conclude that for all $t_0 \in \mathbb{R}_{\ge 0}$, for all $\varepsilon \in (0, \min\{\varepsilon^*, \varepsilon^\#\})$, and for all $x_{20} \in \bar{\mathcal{B}}_r^{n_2}$,

$$|x_{10}| \le c_1 \implies |x_1^\varepsilon(t)| < c_2, \quad \forall t \ge t_0 + T,$$

which completes the proof.

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