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Quasi-LPV Control Design for a Class of Underactuated Mechanical Systems*

Bo Wang, Sergey Nersesov, and Hashem Ashrafiuon

Abstract— **In this work, we present a quasi-linear parameter varying (quasi-LPV) control design approach for a class of underactuated mechanical systems in cascaded normal form. By incorporating linear system theory, robust control techniques, and linear matrix inequality (LMI) tools, a trajectory-based approach is employed to prove semi-global asymptotic stability of the closed-loop systems. The proposed controller can be applied to a wide class of mechanical systems with various degrees of underactuation. The simple structure of the controller facilitates its straightforward implementation for a variety of applications when compared to existing approaches. The controller is implemented for the translational oscillator with a rotational actuator (TORA) system and an underactuated quadrotor model to illustrate its broad applicability and effectiveness.**

Index Terms— **Underactuated systems, stability analysis, quasi-linear parameter variation, linear matrix inequalities.**

I. Introduction

A. Motivation and Related Works

A mechanical system is underactuated if it has fewer number of independent actuators than degrees of freedom to be controlled. Control of underactuated mechanical systems has been extensively studied in control community during the past decades because the numerous practical systems in real life are underactuated including mobile robots, surface vessels, quadrotors, legged robots, etc. Compared with fully actuated systems, underactuated systems enjoy the advantages of simpler structure, less energy consumption, higher flexibility, and lighter weight, etc [1]. Another motivation for studying underactuated systems is fault tolerance when actuators in fully actuated systems fail.

On the other hand, the control design for underactuated systems is far more challenging than for fully actuated systems. Specifically, all fully actuated systems are feedback equivalent to the double-integrator dynamics, which makes the control problems easier to handle. There is an abundance of design methodologies for fully actuated models, see for example, the Slotine-Li controller [2], the PD+ controller [3], sliding mode control methods [4], etc. On the other hand, control of underactuated systems is still an open and interesting problem. Although there is extensive research literature in this field and various special cases are considered, there are relatively few general principles. With fewer number of independent actuators than degrees of freedom, part of the dynamics of underactuated systems cannot be controlled directly by the input. Consequently, the structure of the nonlinear internal coupling between the actuated and unactuated dynamics must be fully considered and utilized in order to achieve the control objective. Hence, comparably diverse design tools are needed for control of underactuated systems to deal with diverse nonlinear internal coupling.

Numerous control methods have been proposed for various kinds of underactuated systems in the literature, including partial feedback linearization [5], interconnection and damping assignment passivity-based control (IDA-PBC) [6], generalized canonical transformations [7], partial stability theory [8], backstepping techniques [9], sliding mode control [10], immersion and invariance control (I&I) [11], to name a few. Readers are referred to the survey paper for more detailed studies about control of underactuated systems [1]. In the seminal work [12], a systematic approach is proposed to transform the coupled underactuated systems obtained from the partial feedback linearization [5] into the uncoupled cascaded normal form, which leads to a classification of underactuated systems based on their structural properties. Later, a first-order sliding mode control design approach is proposed in [13] to stabilize a class of underactuated systems under model uncertainties and disturbances based on the cascaded normal form. In [14], a second-order sliding mode control approach is proposed for stabilization of underactuated systems with non-diminishing disturbances using continuous control inputs. A high-order disturbance observer-based sliding mode controller is proposed in [15] to deal with more general underactuated systems which cannot be described by the cascaded normal form. In [16], a gain-adapting coupling controller is proposed for a class of underactuated mechanical systems to improve transient performance and robustness. While this brief discussion is not intended to be a comprehensive review, it is found that most of controllers in the existing work either have complicated structures or are computationally demanding. Furthermore, some of the existing works impose strict assumptions on the system configuration making them limited and less practical.

B. Main Contributions and Outline

A novel quasi-linear parameter varying (quasi-LPV) control design approach is proposed for a class of underactuated mechanical systems in cascaded normal form with the following main contributions:

1) The proposed controller can be applied to a wide class of underactuated mechanical systems of various degrees of underactuation.

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2) A trajectory-based approach guarantees the semi-global asymptotic stability of the closed-loop systems by incorporating linear system theory, robust control techniques, and linear matrix inequality (LMI) tools.
3) Compared with existing methods, the controller is easy to implement and more practical due to its relatively simple structure which explores the nonlinear coupling between the actuated and unactuated dynamics

The rest of this paper is organized as follows. Preliminaries and problem formulation are given in Section II. Section III presents the controller design and the stability analysis. Illustrative examples are presented in Section IV and the concluding remarks are provided in Section V.

II. Preliminaries and Problem Statement

A. Notations

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space; $\mathbb{R}_{\geq 0}$ denote the set of all non-negative real numbers; $|\cdot|$ denote the Euclidean norm of vectors in $\mathbb{R}^n$; $\mathbb{C}$ denote the set of all complex numbers; and I_n denote the $n \times n$ identity matrix. For $s \in \mathbb{C}$, we let $\mathfrak{Re}(s)$ denote the real part of s. For a real matrix $A \in \mathbb{R}^{n\times n}$, we use $\|A\| = \sup\{|Ax| : |x| = 1\}$ to denote the matrix norm. We omit the arguments of functions when they are clear from the context.

B. Problem Statement

The equations of motion of underactuated mechanical system can be written in the general form

$$\begin{aligned} m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q,\dot{q}) &= 0, \\ m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q,\dot{q}) &= \tau, \end{aligned} \tag{1}$$

where $q_1 \in \mathbb{R}^{n_1}, q_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2$ is the number of degrees of freedom, $q = [q_1^\top, q_2^\top]^\top$ denotes the configuration vector, $(q,\dot{q})$ is the state vector, $\tau \in \mathbb{R}^{n_2}$ is the control input, $m_{ij}(q) \in \mathbb{R}^{n_i\times n_j}$ is the (i, j)-th block-entry of the inertia matrix, and $h_i \in \mathbb{R}^{n_i}$, $i = 1,2$, contain Coriolis, centrifugal and gravity terms. In [12], a systematic approach is proposed to transform the underactuated mechanical system (1) into the following cascaded normal form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_1(x_1,x_2,x_3,x_4), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= f_2(x_1,x_2,x_3,x_4) + g(x_1,x_2,x_3,x_4)u, \end{aligned} \tag{2}$$

where $x_1,x_2 \in \mathbb{R}^{n_1}$ and $x_3,x_4 \in \mathbb{R}^{n_2}$ represent the $n = 2n_1 + 2n_2$ state variables, $u \in \mathbb{R}^{n_2}$ is the control input, $f_1 : \mathbb{R}^n \to \mathbb{R}^{n_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{n_2}$ are smooth vector functions, and $g : \mathbb{R}^n \to \mathbb{R}^{n_2\times n_2}$ is a smooth input matrix. Many underactuated systems can be transformed into the form (2) including the cart-pole system, the translational oscillator with a rotational actuator (TORA) system, surface vessels, the vertical takeoff and landing (VTOL) aircraft, and quadrotors, to name a few [12], [17]. The control objective is to stabilize the origin of the underactuated system (2). We have the following assumptions on system (2).

Assumption 1: $f_1(0,0,0,0) = 0$.

Assumption 2: Either $\partial f_1/\partial x_3$ or $\partial f_1/\partial x_4$ is invertible at the origin. Moreover, the matrix $g(x_1,x_2,x_3,x_4)$ is invertible for all $(x_1,x_2,x_3,x_4) \in \mathbb{R}^n$.

Assumption 3: The function f_1 satisfies

$$|f_1(x_1,x_2,x_3,x_4)| \leq \sum_{i=1}^{4} \rho_i(|x|)|x_i|, \tag{3}$$

where $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ $(i = 1,2,3,4)$ are some globally invertible functions.

Assumption 1 is a necessary condition for the origin to be an equilibrium of the closed-loop system. Assumption 2 requires that the coupling between the unactuated (x_1,x_2)-dynamics and the actuated (x_3,x_4)-dynamics is non-zero at the origin, which is a necessary condition for the (x_1,x_2)-dynamics to be controlled through the coupling. It is noted that Assumption 2 is less conservative than those in [13], [14], which require that either $\partial f_1/\partial x_3$ or $\partial f_1/\partial x_4$ be invertible in the entire $\mathbb{R}^n$. Assumption 3 is a requirement to develop the quasi-LPV controller and is not necessarily satisfied by all the systems under consideration. However, we will show that a further coordinate transformation may yield the function f_1 to satisfy (3), as demonstrated by the examples in Section IV.

C. Quadratic $\mathbb{D}$-Stabilization

A quasi-LPV system is a linear time-varying system whose system matrices are functions of some varying parameters $\delta(\cdot)$, which depend on the state variables, that is,

$$\dot{x}(t) = \mathcal{A}(\delta(x(t)))x(t) + \mathcal{B}(\delta(x(t)))u(t). \tag{4}$$

In some cases, the quasi-LPV description is obtained by rewriting the nonlinear terms in the system equation as the linear terms with respect to the system state variables with time-varying parameters. The quasi-LPV description makes it possible to analyze the nonlinear underactuated system (2) as a linear system. For further information about LPV systems and the classical gain scheduling methods, see [18].

It is well known that for the linear time-varying system $\dot{x} = A(\delta(t))x$, the stability cannot be determined by its eigenvalues. That is, the system matrix $A(\delta(t))$ being Hurwitz for any $\delta(t) \in \Delta$ no longer guarantees uniform asymptotic stability of the system. However, the existence of a common quadratic Lyapunov function for a family of parameter-dependent linear time-invariant systems $\dot{x} = A(\delta)x$ for all $\delta \in \Delta$ guarantees the uniform asymptotic stability of the linear time-varying system, i.e., quadratic stability. We recall some definitions and results from robust control theory. Let $\mathbb{D} \subset \mathbb{C}$ be a symmetrical subset about the real axis. A linear time invariant system $\dot{x} = Ax$ is said to be $\mathbb{D}$-stable if the eigenvalues of $A \in \mathbb{R}^{n\times n}$ satisfy $\lambda_i(A) \in \mathbb{D}$, $i = 1,\ldots,n$. If a linear time-varying system $\dot{x} = A(\delta(t))x$ is quadratic $\mathbb{D}$-stable with $\delta(t) \in \Delta$, then for every $\delta \in \Delta$, the linear parameter-dependent system $\dot{x} = A(\delta)x$ is $\mathbb{D}$-stable. See [19] for more details. We recall the following robust quadratic stabilization result for linear time-varying systems.

Lemma 1 (Quadratic $\mathbb{D}$-stabilization [19]): Consider the quasi-LPV system (4). Assume that $\mathcal{A}(\delta) = \mathcal{A}_0 + \sum_{i=1}^{l} \delta_i \mathcal{A}_i$

and that $\mathcal{B}(\delta) = \mathcal{B}_0 + \sum_{i=1}^{l} \delta_i \mathcal{B}_i$, where $\delta = [\delta_1, \ldots, \delta_l]^\top$, and $\delta_i \in [\delta_i^-, \delta_i^+]$. Let $\mathbb{D} = \{s \in \mathbb{C} : \mathfrak{Re}(s) \in (-\beta, -\alpha)\}$. Then, there exists a state feedback control law $u = \mathcal{K}x$ such that the closed-loop system is quadratic $\mathbb{D}$-stable if and only if there exist a positive definite matrix P and a matrix W satisfying the following set of LMIs:

$$\begin{cases} \mathcal{A}(\delta)P + P\mathcal{A}(\delta)^\top + \mathcal{B}(\delta)W + W^\top \mathcal{B}(\delta) + 2\alpha P < 0, \\ \mathcal{A}(\delta)P + P\mathcal{A}(\delta)^\top + \mathcal{B}(\delta)W + W^\top \mathcal{B}(\delta) + 2\beta P > 0, \end{cases} \tag{5}$$

for any $\delta \in \{(\Delta_1, \ldots, \Delta_l) : \Delta_i = \delta_i^- \text{ or } \delta_i^+, i = 1, \ldots, l\}$. In this case, the gain matrix is given by $\mathcal{K} = WP^{-1}$. □

III. Main Results

In this section, a quasi-LPV based control design approach is proposed for underactuated systems in normal form (2), and the stability analysis for the closed-loop system is given.

A. Controller Design

We study the class of nonlinear systems of the form (2). Assumption 3 provides us a possibility to rewrite the underactuated system (2) into a quasi-LPV form with global bounded varying parameters and to analyze it using linear systems theory. To this end, we assume that there exist a state transformation $\xi = \mathcal{T}_x(x)$ and a feedback transformation $v = \mathcal{T}_u(x, u)$ such that the system (2) is written into the following quasi-LPV form:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & 0 & I_{n_2} \\ \delta_1(\xi) & \delta_2(\xi) & \delta_3(\xi) & \delta_4(\xi) \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{A}(\delta(\xi(t)))} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ I_n \end{bmatrix}}_{\mathcal{B}} v, \tag{6}$$

where $\xi = [\xi_1^\top, \xi_2^\top, \xi_3^\top, \xi_4^\top]^\top \in \mathbb{R}^n$, and $\delta_i(\cdot)$'s are globally bounded varying parameters, i.e., $|\delta_i(\xi)| \in [\delta_i^-, \delta_i^+]$ for all $\xi \in \mathbb{R}^n$ ($i = 1, 2, 3, 4$). If Assumptions 1-3 are satisfied, then the state transformation and the feedback transformation can be simply selected as $\xi_1 = x_1$, $\xi_2 = x_3$, $\xi_3 = x_2$, $\xi_4 = x_4$, and $v = f_2(x_1, x_2, x_3, x_4) + g(x_1, x_2, x_3, x_4)u$. If the Assumption 3 is not satisfied, as discussed above, the coordinate transformation $\mathcal{T}_x(\cdot)$ may be used to rewrite (2) into (6); see the quadrotor problem in Section IV as an example. The varying parameters δ_2 and δ_4 represent the coupling between the unactuated dynamics and the actuated dynamics. It follows from Assumption 2 that either $\delta_2(0)$ or $\delta_4(0)$ is invertible. Note that the quasi-LPV form (6) is reminiscent of the controlled double-integrator dynamics

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{7}$$

Under the linear state feedback $u = -k_1 z_1 - k_2 z_2$, the origin of the closed-loop system is asymptotically stable if and only if $a_{21} - k_1 < 0$ and $a_{22} - k_2 < 0$. This observation is useful in design of controllers for the quasi-LPV system (6). It is also noted that if $\delta_2(0)$ is invertible, then the linearized system of (6) at the origin is controllable since $\operatorname{rank}\{\mathcal{B}, \mathcal{A}(\delta(0))\mathcal{B}, \mathcal{A}(\delta(0))^2\mathcal{B}, \mathcal{A}(\delta(0))^3\mathcal{B}\} = n$. Similarly, if $\delta_4(0)$ is invertible, and either $\delta_1(0)$ or $\delta_3(0)$ is invertible, then linearized system (6) at the origin is controllable. We will discuss these two cases separately in the following context. Recall that if a linear time-invariant system is controllable, then the closed-loop poles can be assigned arbitrarily by state feedback. Furthermore, the controllability of the linearized system implies the existence of linear state feedback that locally stabilizes the origin. We only discuss the following two cases for $n = 1$ for simplicity. Note that the control design and conclusions can be easily generalized to cases of $n > 1$.

B. Stability Analysis

Case 1. ($\delta_2(0) \neq 0$): The condition $\delta_2(0) \neq 0$ implies that the linearized system is controllable, and the eigenvalues of the closed-loop system can be assigned arbitrarily. It follows from Assumption 3 that $\delta_2(\xi) \in [\delta_2^-, \delta_2^+]$ for all $\xi \in \mathbb{R}^4$. If $\delta_2^+ > \delta_2^- > 0$ or $\delta_2^- < \delta_2^+ < 0$ (i.e., $\delta_2(\xi)$ is separated from zero for all $\xi \in \mathbb{R}^4$), then the controllability condition $\operatorname{rank}\{\mathcal{B}, \mathcal{A}(\delta(\xi))\mathcal{B}, \mathcal{A}(\delta(\xi))^2\mathcal{B}, \mathcal{A}(\delta(\xi))^3\mathcal{B}\} = 4$ holds for all $\xi \in \mathbb{R}^4$, which suggests that the stabilization problem for the quasi-LPV system (6) can be easily solved by linear state feedback $v = \mathcal{K}\xi$ via solving the LMIs (5) in Lemma 1. In this case, a common strict quadratic Lyapunov function $V(\xi) = \xi^\top P \xi$ for all $\delta_2 \in [\delta_2^-, \delta_2^+]$ is obtained by solving the LMIs (5). Therefore, we only need to consider the case $\delta_2^- \leq 0 < \delta_2^+$ where linear controllability is lost at $\delta_2 = 0$ and the LMIs (5) may have no solution. Thus, there exists no such a common *quadratic* strict Lyapunov function for all $\delta_2 \in [\delta_2^-, \delta_2^+]$.

Without any loss of generality, we assume that $-\delta_2^- < \delta_2^+$. For each $\delta_2' \in (0, \delta_2^+)$ and for each $\delta_2(\xi) \in [\delta_2', \delta_2^+]$, the quadratic $\mathbb{D}$-stabilization problem is solvable for any $\beta > \alpha > 0$ because the controllability condition holds. Then, for any $\beta > \alpha > 0$, solving the LMIs (5) provides for a quadratic strict Lyapunov function $V(\xi) = \xi^\top P \xi$, and $V(\xi(t))$ decreases exponentially along trajectories as long as $\delta_2(\xi(t))$ belongs the quadratic stable (QS) region, i.e., $\delta_2(\xi(t)) \in [\delta_2', \delta_2^+]$, as shown in Fig. 1. The function $V(\xi(t))$ may increase when $\delta_2(\xi(t))$ belongs the non-QS region. Note that $\delta_2 = 0$ implies that the eigenvalues corresponding to the uncontrollable dynamics have zero real parts. Since eigenvalues are continuous functions of entries of a matrix, the eigenvalues corresponding to the unactuated dynamics may have small positive real parts under small variations of δ_2 in the non-QS region. Therefore, for each compact set centered at the origin, by selecting $\beta > \alpha > 0$ sufficiently large, the Lyapunov function $V(\xi(t))$ converges to zero along trajectories if δ_2 is non-zero definite at the origin.

Case 2. ($\delta_2(0) = 0$, $\delta_4(0) \neq 0$, and either $\delta_1(0) \neq 0$ or $\delta_3(0) \neq 0$): The conditions $\delta_2(0) = 0$, $\delta_4(0) \neq 0$, and either $\delta_1(0) \neq 0$ or $\delta_3(0) \neq 0$ imply that the linearized system is controllable, and the eigenvalues of the closed-loop system can be assigned arbitrarily when $\xi = 0$.

Similarly, the stabilization problem for the quasi-LPV system (6) can be solved by linear state feedback $v = \mathcal{K}\xi$ via solving the LMIs (5) in Lemma 1 if δ_4 is separated from

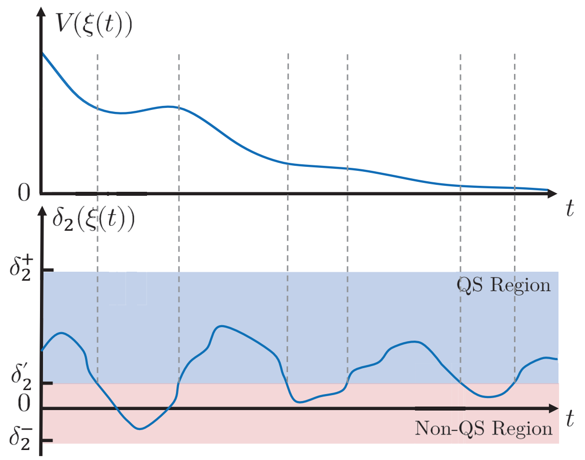

Fig. 1. The response of the displacement z_1, the angle θ_1, and the control input u of the TORA system.

zero. If $\delta_4^- < 0 < \delta_4^+$ but $\max\{|\delta_4^-|, |\delta_4^+|\}$ is small enough, then dividing the parameter space into QS region and non-QS region, for any $\beta > \alpha > 0$, and solving the LMIs (5) gives a quadratic strict Lyapunov function $V(\xi) = \xi^\top P \xi$. Similar to the Case 1, the Lyapunov function decreases along trajectories when δ belongs to the QS region and may increase along trajectories when δ belongs to the non-QS region. Therefore, for each compact set centered at the origin, by selecting sufficiently large $\beta > \alpha > 0$, the Lyapunov function $V(\xi(t))$ converges to zero along trajectories, if δ_4 is non-zero definite at the origin, and either δ_1 or δ_3 is non-zero definite at the origin.

The two cases are summarized into the following theorem with the proof as stated above.

Theorem 1: Consider the quasi-LPV system (6). Assume that one of the following conditions holds: i.) $\delta_2(0)$ is invertible; ii.) $\det\{\delta_2(0)\} = 0$, $\delta_4(0)$ is invertible, and either $\delta_1(0)$ or $\delta_3(0)$ is invertible. For each compact set centered at the origin, by selecting sufficiently large $\beta > \alpha > 0$, under the control law $v = \mathcal{K}\xi$ with $\mathcal{K}$ given by Lemma 1, the origin of closed-loop system is asymptotically stable, and trajectories starting in the compact set converge to the origin as $t \to \infty$.

IV. Illustrative Examples

In this section, the proposed control approach is applied to the TORA system and the underactuated quadrotor system, where simulation results are given to demonstrate the effectiveness of the proposed approach.

A. TORA System

1) Dynamic Model: The TORA system ($n_1 = n_2 = 1$) is represented by the following dynamic equations [20]

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \frac{-z_1 + \varepsilon\theta_2^2 \sin\theta_1}{1 - \varepsilon^2\cos^2\theta_1} - \frac{\varepsilon\cos\theta_1}{1 - \varepsilon^2\cos^2\theta_1} u, \\
\dot{\theta}_1 &= \theta_2, \\
\dot{\theta}_2 &= \frac{\varepsilon\cos\theta_1(z_1 - \varepsilon\theta_2^2\sin\theta_1)}{1 - \varepsilon^2\cos^2\theta_1} + \frac{1}{1 - \varepsilon^2\cos^2\theta_1} u,
\end{aligned} \tag{8}$$

where z_1 and z_2 denote the displacement and velocity of the platform, θ_1 and θ_2 denote angle and angular velocity of the rotor, u is the control torque applied to the rotor, and the coefficient ε depends on the physical parameters of the system. A typical value for ε is 0.1.

2) State and Feedback Transformation: Employing the following state and feedback transformation:

$$\begin{aligned}
x_1 &= z_1 + \varepsilon\sin\theta_1, \qquad x_3 = \theta_1, \\
x_2 &= z_2 + \varepsilon\theta_2\cos\theta_1, \qquad x_4 = \theta_2, \\
v &= \frac{\varepsilon\cos\theta_1(z_1 - \varepsilon\theta_2^2\sin\theta_1) + u}{1 - \varepsilon^2\cos^2\theta_1},
\end{aligned} \tag{9}$$

the TORA system (8) can be written into the following cascaded normal form:

$$\begin{aligned}
\dot{x}_1 &= x_2, \qquad \dot{x}_3 = x_4, \\
\dot{x}_2 &= -x_1 + \varepsilon\sin x_3, \qquad \dot{x}_4 = v,
\end{aligned} \tag{10}$$

It can be easily verified that the system (10) satisfies Assumptions 1-3. Under a further coordinate transformation $\xi_1 = x_1$, $\xi_2 = x_3$, $\xi_3 = x_2$, and $\xi_4 = x_4$, the system can be written as a quasi-LPV system

$$\dot{\xi} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & \varepsilon\delta(\theta_1(t)) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{A}(\delta(\cdot))} \xi + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathcal{B}} v, \tag{11}$$

where $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]^\top$, and $\delta(\theta_1) = \sin(\theta_1)/\theta_1$. Note that the function $\delta(\cdot)$ is smooth and bounded on its domain, i.e., $\delta(\theta_1) \in (-0.22, 1]$ for all $\theta_1 \in \mathbb{R}$.

3) Simulation Results: The initial condition for the TORA system is $(z_1(0), z_2(0), \theta_1(0), \theta_2(0)) = (1, 0, 0, 0)$. The goal of the controller is to stabilize the frictionless oscillator as quickly as possible using the rotational actuator. We choose the QS region $[\delta', \delta^+] = [0.022, 0.1]$. The parameters in LMIs (5) are selected as $\alpha = 0.8$ and $\beta = 1.5$. The control gains are obtained as $k_1 = -3.784, k_2 = -5.673, k_3 = -8.346, k_4 = -3.013$. The simulation results of the TORA system are shown in Fig. 2. The displacement trajectory, the angle trajectory, and the control input demonstrate practical profiles and converge to zero in about only 30 seconds.

B. Underactuated Quadrotor System

1) Dynamic Model: The dynamic model of an underactuated quadrotor ($n_1 = 2$, $n_2 = 4$) is given by the following equations in global coordinates [13]

$$\begin{aligned}
\ddot{x} &= u_1(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi), \\
\ddot{y} &= u_1(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi), \\
\ddot{z} &= u_1\cos\phi\cos\theta - g, \\
\ddot{\phi} &= u_2, \\
\ddot{\theta} &= u_3, \\
\ddot{\psi} &= u_4,
\end{aligned} \tag{12}$$

where (x, y, z) represents the position of the center of mass, (ϕ, θ, ψ) represents the Euler angles, u_i's are the feedback

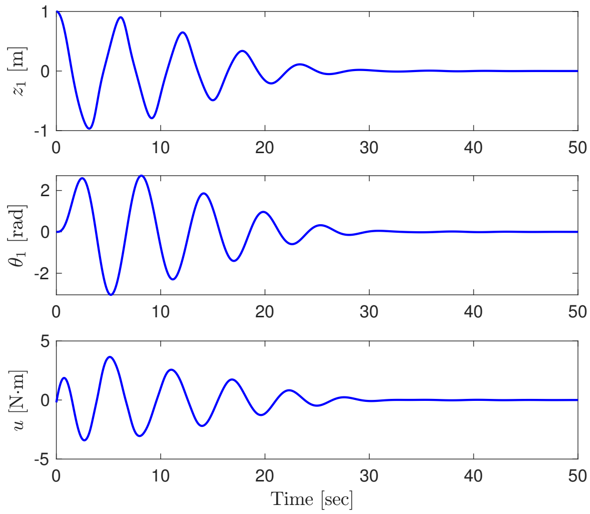

Fig. 2. Time history of the mass displacement z_1, pendulum angle θ_1, and the quasi-LPV designed control input u of the TORA system.

linearized control inputs ($i = 1,2,3,4$), and g is the acceleration of gravity. To avoid singularities associated with Euler angle, we restrict the angles to satisfy $|\phi| < \pi/2$ and $|\theta| < \pi/2$. Let us consider the trajectory tracking problem for the quadrotor, where the desired trajectory is given by $(x_d(t), y_d(t), z_d(t), \psi_d(t))$ for $t \geq 0$. The control objective is to design a state feedback controller (u_1, u_2, u_3, u_4) such that i.) all the signals in the closed-loop system are bounded for all $t \geq 0$; ii.) the tracking error converges to zero, i.e.,

$$\lim_{t\to\infty} |(x(t), y(t), z(t), \psi(t)) - (x_d(t), y_d(t), z_d(t), \psi_d(t))| = 0.$$

Because the fully actuated (z, ψ)-dynamics can be directly controlled by inputs (u_1, u_4), we select a simple PD control law for the second-order (z, ψ)-system, i.e.,

$$u_1 = \frac{1}{\cos\phi\cos\theta}\left[\ddot{z}_d - k_{zp}(z - z_d) - k_{zd}(\dot{z} - \dot{z}_d)\right], \tag{13}$$

$$u_4 = \ddot{\psi}_d - k_{\psi p}(\psi - \psi_d) - k_{\psi d}(\dot{\psi} - \dot{\psi}_d), \tag{14}$$

where $k_{zp}, k_{zd}, k_{\psi p}$ and $k_{\psi d}$ are positive control gains. Note that since $|\phi| < \pi/2$ and $|\theta| < \pi/2$, the control law (13) is well defined, and the error system for (z, ψ)-dynamics is given by

$$(\ddot{z} - \ddot{z}_d) = -k_{zp}(z - z_d) - k_{zd}(\dot{z} - \dot{z}_d), \tag{15}$$

$$(\ddot{\psi} - \ddot{\psi}_d) = -k_{\psi p}(\psi - \psi_d) - k_{\psi d}(\dot{\psi} - \dot{\psi}_d), \tag{16}$$

which implies both $z(t) - z_d(t)) \to 0$ and $\psi(t) - \psi_d(t) \to 0$ exponentially as $t \to \infty$. We will focus on the underactuated (x, y, ϕ, θ)-dynamics and treat $u_1(t)$ and $\psi(t)$ as bounded time functions in the (x, y, ϕ, θ)-dynamics due to the exponential convergence.

2) State and Feedback Transformation: Define $x_1 = [x, y]^\top$, $x_2 = [\dot{x}, \dot{y}]^\top$, $x_3 = [\phi, \theta]^\top$, and $x_4 = [\dot{\phi}, \dot{\theta}]^\top$, The (x, y, ϕ, θ)-dynamics are transformed into the cascaded normal form (2) with

$$f_1 = \begin{bmatrix} u_1(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\ u_1(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \end{bmatrix},$$

$f_2 = 0$, and $g = I_2$. Note that the transformed system does not satisfy the Assumption 3. However, under a further state transformation $\xi_1 = x_1$, $\xi_2 = x_3$, $\xi_3 = [\theta, -\phi]^\top$, and $\xi_4 = [\dot{\theta}, -\dot{\phi}]^\top$, Assumption 3 is satisfied and the transformed dynamics can be written into the following quasi-LPV form

$$\dot{\xi} = \underbrace{\begin{bmatrix} 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & \mathcal{A}_{32}(\delta(\cdot)) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{A}(\delta(\cdot))} \xi + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2 \end{bmatrix}}_{\mathcal{B}} v, \tag{17}$$

where $\xi = [\xi_1^\top, \xi_2^\top, \xi_3^\top, \xi_4^\top]^\top$, $v = [u_3, -u_2]$, and

$$\mathcal{A}_{32} = u_1 \begin{bmatrix} \frac{\sin\theta}{\theta}\cos\phi\cos\psi & -\frac{\sin\phi}{\phi}\sin\psi \\ \frac{\sin\theta}{\theta}\cos\phi\sin\psi & \frac{\sin\phi}{\phi}\cos\psi \end{bmatrix}.$$

3) Controller Design: Since the transformed system satisfies the conditions of Theorem 1, a quasi-LPV control design can be applied to *stabilize* the quadrotor system (17). Then, the closed-loop error dynamics can be written into the quasi-LPV form $\dot{\xi} = \mathcal{A}(\xi(t))\xi + \mathcal{B}\mathcal{K}\xi + \mathcal{B}w = \mathcal{A}_c\xi + \mathcal{B}w$, where the origin of $\dot{\xi} = \mathcal{A}_c\xi$ is asymptotically stable and the additional input w is used to solve the *tracking* problem. We rewrite the desired trajectory into a exo-system in the same quasi-LPV form, i.e., $\dot{\xi}_d = \mathcal{A}_c\xi_d + \mathcal{B}_c w_d$. Define the tracking error $\xi_e(t) := \xi(t) - \xi_d(t)$ and consequently the error dynamics $\dot{\xi}_e = \mathcal{A}_c\xi_e + \mathcal{B}_c(w - w_d)$. Next, simply choosing $w = w_d$ yields an asymptotically stable error dynamics, which implies the tracking error tends to zero as $t \to \infty$.

4) Simulation Results: The initial condition for the quadrotor system is $(x(0), y(0), z(0), \phi(0), \theta(0), \psi(0)) = (10, -5, 0, 0, 0, 0)$. Assume that the desired trajectory is set to $x_d(t) = 2 + \cos(t)$, $y_d(t) = \sin(t)$, $z_d(t) = 2$, and $\psi_d(t) = \cos(t)$ for all $t \geq 0$. The control parameters are selected as $k_1 = -1, k_2 = -15, k_3 = -3, k_4 = -3$. The simulation results of the quadrotor system are shown in Figs. 3-5. Figure 3 shows the actual position and yaw angle state trajectories reaching and following the desired state trajectories within 10 seconds. Figure 4 shows the quadrotor path smoothly reaching and following the desired circular path. The well-behaved trajectories of the four control inputs are shown to in Fig. 5.

V. Conclusions

In this work, we presented a quasi-LPV control design approach for a class of underactuated mechanical systems in cascaded normal form. By incorporating linear system theory, robust control techniques, and LMI tools, a trajectory-based stability argument was employed to prove the semi-global asymptotic stability for the closed-loop system. The controller was shown to be applicable to a broad class of underactuated mechanical systems and possess a simple structure compared to the existing approaches in the

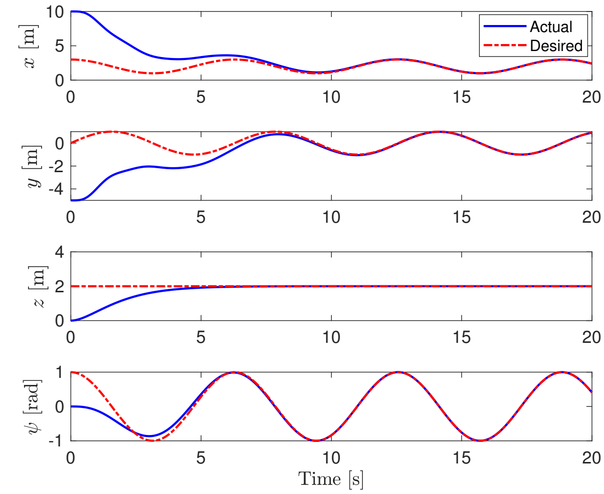

Fig. 3. Time history of the position and yaw angle states for the underactuated quadrotor system.

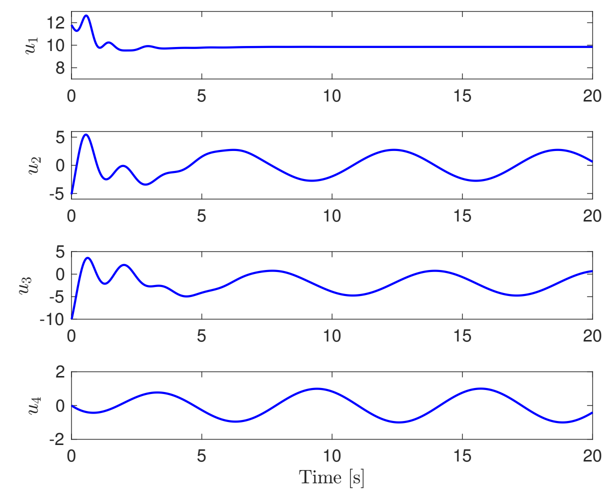

Fig. 5. Time trajectories of the four control inputs of the quadrotor system.

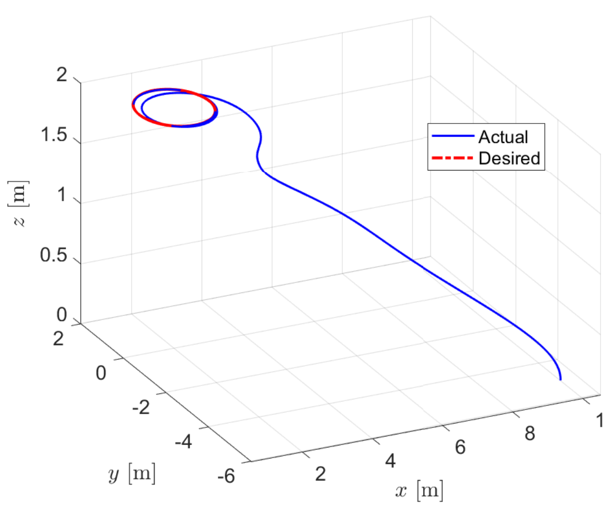

Fig. 4. The actual and desired paths of the quadrotor in the 3D space.

literature. Two typical examples of underactuated systems, the TORA system and quadrotor drone, were presented and simulated to demonstrate the effectiveness of the proposed approach.

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