Extremum Seeking Control for Antenna Pointing via Symmetric Product Approximation

Bo Wang[1] Hashem Ashrafiuon[2] Sergey G. Nersesov[2]

[1]Department of Mechanical Engineering
The City College of New York, CUNY

[2]Department of Mechanical Engineering
Villanova University

2025 Modeling, Estimation and Control Conference (MECC)
Pittsburgh, PA, October 2025

Antenna pointing system

Problem: make antenna to autonomously adjust its direction to maximize the received signal strength **without reference measurements**

Traditional Control Methods
(often assume the reference attitude is known)

- **Proportional-Integral (PI)**
- **LQG**
- **H-infinity**
- **…**

Reference attitude usually is unknown!

Antenna pointing system

Problem: make antenna to autonomously adjust its direction to maximize the received signal strength **without reference measurements**

$$\omega_x = -\dot{\theta}_2 \sin\theta_1, \quad \omega_y = \dot{\theta}_1, \quad \omega_z = \dot{\theta}_2 \cos\theta_1.$$

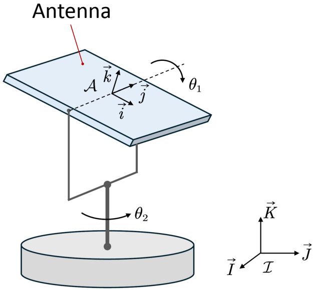
$$I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z = -\dot{\theta}_2 \sin\theta_1$$

$$I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x = \dot{\theta}_1$$

$$I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y = \dot{\theta}_2 \cos\theta_1$$

Euler-Lagrange eqn:

$$\boxed{\begin{aligned} \dot{\theta} &= J(\theta)\omega \\ I\dot{\omega} + C(\omega)\omega + D\omega &= J(\theta)^\top \tau \end{aligned}}$$



Antenna pointing system

Problem: make antenna to autonomously adjust its direction to maximize the received signal strength **without angular measurements**

Extremum Seeking Control (ESC)
(no need to know the reference)

- **Real-time optimization**
- **Model-free**
- **Applied to dynamical systems**
- **…**

$$\omega_x = -\dot{\theta}_2 \sin\theta_1, \quad \omega_y = \dot{\theta}_1, \quad \omega_z = \dot{\theta}_2 \cos\theta_1.$$

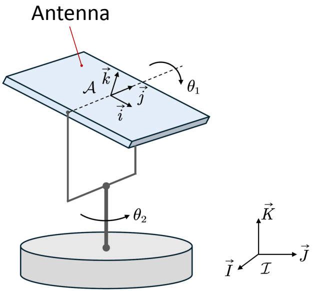
$$I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z = -\dot{\theta}_2 \sin\theta_1$$

$$I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x = \dot{\theta}_1$$

$$I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y = \dot{\theta}_2 \cos\theta_1$$

Euler-Lagrange eqn:

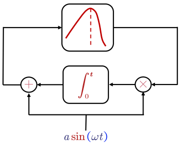
$$\boxed{\begin{aligned} \dot{\theta} &= J(\theta)\omega \\ I\dot{\omega} + C(\omega)\omega + D\omega &= J(\theta)^\top \tau \end{aligned}}$$

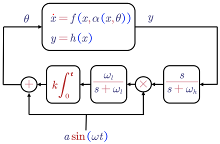


Extremum seeking control (ESC)

Method: Symmetric product approximation

[Classical averaging-based ESC]





[Lie bracket-based ESC]

$$\dot{x} = \alpha\sqrt{\omega}\cos(\omega t) + f(x)\sqrt{\omega}\sin(\omega t)$$

$$\dot{\bar{x}} = \frac{1}{2}[\alpha, f(\bar{x})](\bar{x}) = \frac{\alpha}{2}\nabla f(\bar{x})$$

[Symmetric product approximation-based ESC] applies to *mechanical systems*

$$\ddot{q} = R_0(q) + R(q)\dot{q} + u_\omega(t)F(t,q)$$

$$\ddot{\bar{q}} = R_0(\bar{q}) + R(\bar{q})\dot{\bar{q}} - \frac{1}{4}\langle F : F\rangle(t,\bar{q})$$

Signal assumptions

Measurement: receiving power $p(\theta)$. Want to $\max \; p(\theta)$

Optimization problem: $\min h(\theta) := -p(\theta) + p_0$

[Measurements $h(\theta(t)) > 0$]:

(A1) $\exists \, ! \; \theta_* := [\theta_{1*} \; \theta_{2*}]^\top$ s.t.

$$\frac{\partial h}{\partial \theta}(\theta_*) = 0, \quad \text{and} \quad \frac{\partial h}{\partial \theta}(\theta) \neq 0, \quad \forall \theta \neq \theta_*.$$

(A2) h is in the separable form

$$h(\theta) = h_1(\theta_1) + h_2(\theta_2).$$

(A3) $\exists \; h_M > 0$ such that

$$\left\| \frac{\partial^2 h}{\partial \theta^2}(\theta) \right\| \leq h_M, \quad \forall \theta.$$

[Rmk 1] Both the extremum θ_* and the gradient $\dfrac{\partial h}{\partial \theta}(\theta)$ are unknown.

Control law

[System dynamics]: Euler-Lagrangian equation

$$\dot{\theta} = J(\theta)\omega$$
$$I\dot{\omega} + C(\omega)\omega + D\omega = J(\theta)^{\top}\tau$$

[Control law]: high-frequency high-amplitude

$$\boxed{\begin{aligned} \tau_1 &= \frac{k_1}{\varepsilon} u_1\left(\frac{t}{\varepsilon}\right) h(\theta) \\ \tau_2 &= \frac{k_2}{\varepsilon} u_2\left(\frac{t}{\varepsilon}\right) h(\theta) \end{aligned}}$$

Orthogonality cond: $$\int_0^T U_i(\tau)U_j(\tau)\mathrm{d}\tau = \begin{cases} \frac{T}{2} & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

[Closed-loop system]

$$\begin{aligned} \dot{\theta} &= J(\theta)\omega \\ I\dot{\omega} + C(\omega)\omega + D\omega &= J(\theta)^{\top}\left[\frac{h(\theta)}{\varepsilon} K u\left(\frac{t}{\varepsilon}\right)\right] \end{aligned}$$

Averaging analysis

[Closed-loop system]

$$\dot{\theta} = J(\theta)\omega$$
$$I\dot{\omega} + C(\omega)\omega + D\omega = J(\theta)^\top \left[\frac{h(\theta)}{\varepsilon} K u\left(\frac{t}{\varepsilon}\right)\right]$$

[Averged system]: symmetric product system

$$\dot{\bar{\theta}} = J(\bar{\theta})\bar{\omega}$$
$$I\dot{\bar{\omega}} + C(\bar{\omega})\bar{\omega} + D\bar{\omega} = J(\bar{\theta})^\top \Lambda(\bar{\theta}_1)\left[-\frac{1}{2}\frac{\partial h}{\partial \theta}^\top(\bar{\theta})h(\bar{\theta})\right]$$

where

$$\Lambda(\bar{\theta}_1) := \begin{bmatrix} k_1^2 I_y^{-1} & 0 \\ 0 & k_2^2 r(\bar{\theta}_1) \end{bmatrix}, \quad r(\theta_1) := I_x^{-1} s_{\theta_1}^2 + I_z^{-1} c_{\theta_1}^2$$

Theorem (Symmetric product approximation)

If [Averaged system] is uniformly asymptotically stable, then [Closed-loop system] is practically uniformly asymptotically stable.

Stability analysis: Overview

[Averged system]: symmetric product system

$$\dot{\bar{\theta}} = J(\bar{\theta})\bar{\omega}$$

$$I\dot{\bar{\omega}} + C(\bar{\omega})\bar{\omega} + D\bar{\omega} = J(\bar{\theta})^\top \Lambda(\bar{\theta}_1)\left[-\frac{1}{2}\frac{\partial h}{\partial \theta}^\top(\bar{\theta})h(\bar{\theta})\right]$$

where

$$\Lambda(\bar{\theta}_1) := \begin{bmatrix} k_1^2 I_y^{-1} & 0 \\ 0 & k_2^2 r(\bar{\theta}_1) \end{bmatrix}, \quad r(\theta_1) := I_x^{-1} s_{\theta_1}^2 + I_z^{-1} c_{\theta_1}^2$$

[Rmk 2]: If $\Lambda(\cdot)$ is a constant matrix, then **[Averged system]** is reminiscent of a *gradient system* (i.e., a Lagrangian system under PD control)

e.g., mass-spring-damper: $\ddot{x} = -k_d\dot{x} - k_p(x - x_d)$

[Train of thought]:

» "Frozen dynamics"

» strict Lyapunov function

» robustness analysis (slowly time-varying system)

Stability analysis: Frozen dynamics

["Frozen dynamics"]:

$$\begin{aligned}\dot{\bar{\theta}} &= J(\bar{\theta})\bar{\omega} \\ I\dot{\bar{\omega}} + C(\bar{\omega})\bar{\omega} + D\bar{\omega} &= J(\bar{\theta})^\top \bar{\Lambda}\left[-\frac{1}{2}\frac{\partial h}{\partial \theta}^\top(\bar{\theta})h(\bar{\theta})\right]\end{aligned}$$

[Weak Lyapunov function]

$$V_1(\bar{\theta}-\theta_*, \bar{\omega}) := \frac{1}{2}\bar{\omega}^\top I\bar{\omega} + \frac{1}{4}\bar{h}(\bar{\theta})^2 - \frac{1}{4}\bar{h}(\theta_*)^2 > 0$$

$$\begin{aligned}\dot{V}_1 &= \bar{\omega}^\top\left[-C(\bar{\omega})\bar{\omega} - D\bar{\omega} + J(\bar{\theta})^\top\bar{\Lambda}\left(-\frac{1}{2}\frac{\partial h}{\partial \theta}^\top(\bar{\theta})h(\bar{\theta})\right)\right] + \frac{1}{2}\bar{h}(\bar{\theta})\frac{\partial \bar{h}}{\partial \theta}(\bar{\theta})J(\bar{\theta})\bar{\omega} \\ &= -\bar{\omega}^\top D\bar{\omega} \leq 0\end{aligned}$$

[Rmk 3]: Asymptotic stability of ["Frozen dynamics"] comes from by verifying the LaSalle's condition on the set $\{\dot{V}_1 = 0\}$

Stability analysis: Strict Lyapunov function

Theorem (Strict Lyapunov function)

$$V_\lambda(\bar{\theta} - \theta_*, \bar{\omega}) := V_2(\bar{\theta}, \bar{\omega}) + P_3(V_1(\bar{\theta} - \theta_*, \bar{\omega}))$$

is a strict Lyapunov function for the ["Frozen dynamics"].

$$V_1(\bar{\theta} - \theta_*, \bar{\omega}) := \frac{1}{2}\bar{\omega}^\top I \bar{\omega} + \frac{1}{4}\bar{h}(\bar{\theta})^2 - \frac{1}{4}\bar{h}(\theta_*)^2.$$

$$V_2(\bar{\theta}, \bar{\omega}) := \frac{\partial \bar{h}}{\partial \theta}(\bar{\theta}) J(\bar{\theta}) I \bar{\omega},$$

$$P_0 := \max\left\{ \frac{12 I_M^2}{I_m}, \frac{4\bar{h}_M^2}{\bar{h}(\theta_*)} \lambda_{\min}^{-1}\left[\frac{\partial^2 \bar{h}}{\partial \theta^2}(\theta_*)\right] \right\}.$$

$$P_1(l) := \sqrt{6} d_M + 8\sqrt{3l}\frac{I_M}{\sqrt{I_m}},$$

$$P_2(l) := 6\bar{h}_M I_M + \frac{1}{h(\theta_*)} P_1(l)^2$$

$$P_3(l) := \frac{1}{d_m}\int_0^l P_2(m)\mathrm{d}m + P_0 l,$$

Stability analysis: Robustness

Proposition (UAS of averaged system)

Consider the equilibrium point $(\bar{\theta}, \bar{\omega}) = (\theta_*, 0)$. Assume that $k_1 > 0$. Then, there exists a positive constant $\bar{k}_2 > 0$ such that for all $k_2 \in (0, \bar{k}_2]$, the equilibrium point $(\bar{\theta}, \bar{\omega}) = (\theta_*, 0)$ is uniformly asymptotically stable.

[Proof]: Rewrite the averaged system as a slowly time-varying system

$$\begin{aligned} \dot{\bar{\theta}} &= J(\bar{\theta})\bar{\omega} \\ I\dot{\bar{\omega}} + C(\bar{\omega})\bar{\omega} + D\bar{\omega} &= J(\bar{\theta})^\top \bar{\Lambda}(t) \left[-\frac{1}{2} \frac{\partial h}{\partial \theta}^\top (\bar{\theta}) h(\bar{\theta}) \right] \end{aligned}$$

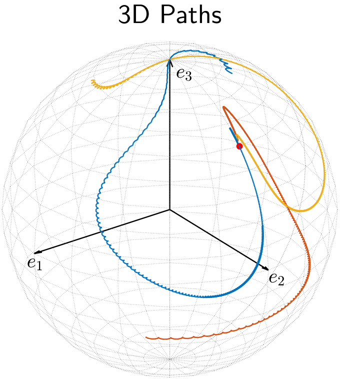
Lyapunov analysis:

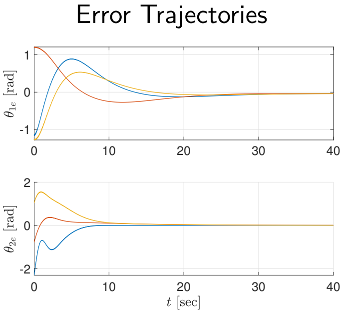
$$\dot{V}_\lambda|_{\text{avg}} = \dot{V}_\lambda|_{\text{frozen}} + \frac{\partial V_\lambda}{\partial \lambda_2} \dot{\lambda}_2(t)$$

The second term can be made arbitrarily small by choosing a sufficiently small control parameter k_2.

Illustrative examples

Main result: Practically uniformly asymptotically stable





To prove further:

- B. Wang, S. Nersesov, H. Ashrafiuon, P. Naseradinmousavi, and M. Krstic, “Underactuated Source Seeking by Surge Force Tuning: Theory and Boat Experiments,” *IEEE Transactions on Control Systems Technology*, Volume 31, Issue 4, July 2023, pp. 1649-1662.
- B. Wang, “Semi-Global Nonholonomic Source Seeking by Torque Tuning," *arXiv*, October 2025. (In preparation)

Thank you!

bwang1@ccny.cuny.edu

https://bwang-ccny.github.io/