ME 41100: System Dynamics and Control
Lecture 8: Root Locus Analysis and Control Design

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Learning Objectives. By the end of this lecture, you should be able to:

- explain why closed-loop poles move as the loop gain varies;
- derive the root-locus angle and magnitude conditions from the characteristic equation;
- sketch a positive-gain root locus using open-loop poles, zeros, real-axis rules, and asymptotes;
- locate breakaway points and imaginary-axis crossings and determine stable gain ranges;
- select a feedback gain from damping-ratio and transient-response specifications;
- verify analytic predictions using MATLAB, distinguishing second-order formulas from high-order approximations.

1 Motivation: From PID to Root Locus

Lecture 5 linked pole locations to transient response. Lecture 6 used the Routh–Hurwitz criterion to test closed-loop stability. Lecture 7 showed that changing P, PI, PD, and PID gains changes the closed-loop characteristic equation. We now study *how* the closed-loop poles move when a gain changes continuously.

Central design question. As the feedback gain $K \geq 0$ increases, where do the closed-loop poles move, and which gain gives a stable response with acceptable transient performance?

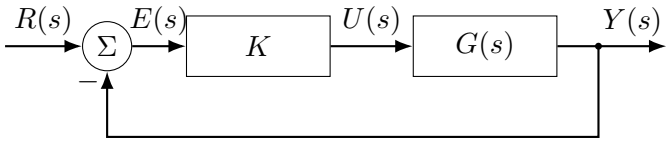


Figure 1: Unity negative feedback with adjustable nonnegative gain K.

With zero initial conditions, let the rational open-loop *shape* be

$$G(s) = \frac{N(s)}{D(s)}, \qquad L(s) = KG(s),$$

where $G(s)$ contains the fixed plant and controller dynamics, while $K \geq 0$ is the adjustable scalar gain. We assume that N and D have no common factors. Then

$$T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KN(s)}{D(s) + KN(s)}, \qquad \boxed{\Delta(s; K) = D(s) + KN(s) = 0.} \tag{1}$$

The *root locus* is the set of roots of $\Delta(s; K)$ in the complex s-plane as K varies from 0 toward $+\infty$.

Do not confuse poles. Open-loop poles and zeros determine the root-locus geometry. *Closed-loop poles* are the moving points on the locus; a chosen gain K selects their locations.

Example: A first look: a second-order closed loop Let $G(s) = 1/[s(s+4)]$. Then

$$\Delta(s;K) = s^2 + 4s + K, \quad s_{1,2} = -2 \pm \sqrt{4-K}.$$

For $0 < K < 4$ the poles are real; at $K = 4$ they meet at $s = -2$; for $K > 4$ they form a complex-conjugate pair $-2 \pm j\sqrt{K-4}$. Increasing K does *not* move their real part farther left once $K > 4$.

Question for Discussion

If K becomes large, does the closed loop always become faster? Does it necessarily have less overshoot?

2 Deriving the Root-Locus Conditions

A point s belongs to a positive-gain root locus if there exists a $K > 0$ for which

$$1 + KG(s) = 0 \quad \Longleftrightarrow \quad KG(s) = -1.$$

Write $G(s) = |G(s)|e^{j\angle G(s)}$. Since $K > 0$ has zero phase, matching the magnitude and angle of -1 yields

$$\boxed{\angle G(s) = (2\ell+1)180^\circ, \quad \ell \in \mathbb{Z}} \qquad \text{(angle condition)} \tag{2}$$

$$\boxed{K = \frac{1}{|G(s)|}} \qquad \text{(magnitude condition).} \tag{3}$$

If $G(s) = c\prod_{j=1}^{m}(s - z_j)/\prod_{i=1}^{n}(s - p_i)$ with $c > 0$, the angle condition becomes

$$\sum_{j=1}^{m} \angle(s - z_j) - \sum_{i=1}^{n} \angle(s - p_i) = (2\ell+1)180^\circ.$$

Interpretation: the angle condition says *where* poles may lie; the magnitude condition gives the required gain *at that point*. For a negative-gain locus the angle rule changes; throughout this lecture we use $K \geq 0$.

Example: Test a candidate point and find its gain For

$$G(s) = \frac{1}{s(s+4)},$$

consider the candidate closed-loop pole $s_d = -2 + 2j$. The vectors from the two open-loop poles to s_d have angles

$$\angle(s_d) = 135^\circ, \qquad \angle(s_d + 4) = 45^\circ.$$

Hence

$$\angle G(s_d) = -135^\circ - 45^\circ = -180^\circ,$$

so s_d satisfies the angle condition and lies on the positive-gain root locus. The magnitude condition then gives

$$K = \frac{1}{|G(s_d)|} = |s_d|\,|s_d + 4| = \sqrt{8}\sqrt{8} = \boxed{8}.$$

Indeed, $s^2 + 4s + 8 = 0$ has roots $-2 \pm 2j$. Thus the angle condition tests a desired location, while the magnitude condition selects the corresponding gain.

3 Basic Root-Locus Construction Rules

Assume a proper rational $G(s)$, $n \geq m$, with n open-loop poles and m open-loop zeros, counted with multiplicity. The rules below describe the standard positive-gain locus.

Rule 1. Branches. There are n branches (the degree of the generic characteristic polynomial).

Rule 2. Starting and ending points. As $K \to 0^+$, branches begin at open-loop poles. As $K \to \infty$, m branches approach open-loop zeros, and $n - m$ branches go to infinity.

Rule 3. Conjugate symmetry. With real polynomial coefficients, the locus is symmetric about the real axis.

Rule 4. Real-axis test. A point on the real axis not at a pole or zero belongs to the positive-gain locus when the total number of *real* open-loop poles and zeros to its right is odd (counting multiplicities).

Rule 5. Asymptotes. When $n > m$, the $n-m$ asymptote angles and their common centroid are

$$\boxed{\theta_q = \frac{(2q+1)180^\circ}{n-m},\ q = 0, \ldots, n-m-1, \qquad \sigma_a = \frac{\sum_i p_i - \sum_j z_j}{n-m}.} \tag{4}$$

The centroid is real for real-coefficient polynomials, since complex roots occur in conjugate pairs.

Sketching strategy. Mark poles with $\times$ and zeros with $\circ$; find real-axis segments; draw asymptotes; then determine breakaway points and imaginary-axis crossings. Do not infer stability from asymptotes alone.

4 Complete Worked Sketch: A Third-Order System

Example: Root locus of $G(s) = 1/[s(s+2)(s+4)]$ For a unity-feedback loop with $K \geq 0$,

$$1 + \frac{K}{s(s+2)(s+4)} = 0 \quad \Longleftrightarrow \quad \boxed{s^3 + 6s^2 + 8s + K = 0.} \tag{5}$$

Step 1: poles and zeros. Open-loop poles are $0, -2, -4$; there are no zeros. Thus there are three branches, all ending at infinity.

Step 2: real-axis segments. Counting real poles and zeros to the right gives

$$\boxed{(-\infty, -4) \quad \text{and} \quad (-2, 0).}$$

Step 3: asymptotes. With $n - m = 3$,

$$\boxed{\sigma_a = \frac{0 - 2 - 4}{3} = -2, \qquad \theta_q = 60^\circ, 180^\circ, 300^\circ.}$$

Two branches leave the real axis as a complex-conjugate pair, while the third continues toward $-\infty$.

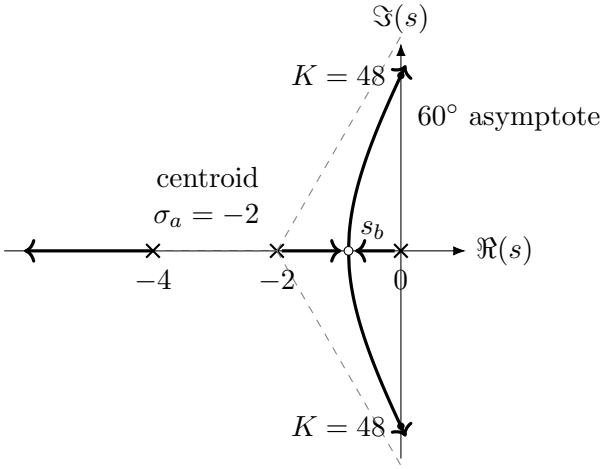


Figure 2: Root locus of $G(s) = 1/[s(s+2)(s+4)]$. The axes are truncated; the complex branches continue toward the $\pm 60^\circ$ asymptotes.

Question for Discussion

Why is $(-4, -2)$ *not* a positive-gain root-locus segment? Apply the odd-number rule to $s = -3$.

5 Breakaway Points and Stability Boundaries

5.1 Finding Breakaway or Break-In Candidates

On a real-axis portion of the locus, the characteristic equation gives the gain as a function of location:

$$K(s) = -\frac{D(s)}{N(s)}.$$

At a real multiple closed-loop root (a typical breakaway/break-in point), a necessary condition is

$$\boxed{\frac{dK(s)}{ds} = 0.} \tag{6}$$

Important: this condition produces *candidates*, not guaranteed locus points. Check $K > 0$, the real-axis rule, and the characteristic equation; special cases need additional care.

For Eq. (5),

$$K(s) = -s(s+2)(s+4), \qquad \frac{dK}{ds} = -(3s^2 + 12s + 8).$$

The stationary points are

$$s = -2 \pm \frac{2\sqrt{3}}{3} \approx -0.8453, \ -3.1547.$$

Only the point in $(-2, 0)$ belongs to the positive-gain root locus:

$$\boxed{s_b = -2 + \frac{2\sqrt{3}}{3} \approx -0.8453.} \tag{7}$$

Its gain is $K_b = 16\sqrt{3}/9 \approx 3.0792$.

5.2 Imaginary-Axis Crossing via Routh–Hurwitz

For the cubic polynomial in Eq. (5), the Routh table is

s^3	1	8
s^2	6	K
s^1	$(48 - K)/6$	0
s^0	K	

All first-column entries are strictly positive exactly when

$$\boxed{0 < K < 48.} \tag{8}$$

At $K = 48$,

$$s^3 + 6s^2 + 8s + 48 = (s + 6)(s^2 + 8),$$

so the crossing is at

$$\boxed{s = \pm j\sqrt{8}.} \tag{9}$$

At $K = 0$, the system also has a pole at the origin and is not asymptotically stable. At $K = 48$, the conjugate imaginary-axis poles prevent asymptotic stability; for $K > 48$ there are two right-half-plane poles.

Connect to Lecture 6. Root locus *visualizes* where instability begins; the Routh–Hurwitz test finds the stability interval algebraically. Use both as consistency checks.

6 From Root Locus to Transient-Response Design

For the standard underdamped second-order polynomial

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad 0 < \zeta < 1, \tag{10}$$

the closed-loop poles are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}.$$

Their location links pole geometry to the damping ratio ζ and natural frequency ω_n. For a *standard second-order response without extra zeros*,

$$M_p = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}\%, \qquad T_s \approx \frac{4}{\zeta\omega_n} \quad (2\% \text{ settling criterion}). \tag{11}$$

For higher-order systems, these formulas are *approximations* that require a justified dominant-pole assumption; zeros and additional modes can significantly change the actual transient.

Example: Choose the gain to achieve $\zeta = 0.5$ Consider again

$$G(s) = \frac{1}{s(s+4)}, \qquad \Delta(s;K) = s^2 + 4s + K.$$

Comparing to Eq. (10) gives

$$2\zeta\omega_n = 4, \qquad \omega_n^2 = K \quad \Longrightarrow \quad \boxed{\zeta = \frac{2}{\sqrt{K}}, \quad \omega_n = \sqrt{K}.}$$

For $\zeta = 0.5$,

$$\boxed{K = 16, \qquad s_{1,2} = -2 \pm j\sqrt{12}.} \tag{12}$$

Geometrically, the constant-damping-ratio lines for $\zeta = 0.5$ make angles of 120° and 240° from the positive real axis. Their intersections with the root locus identify the desired pole pair.

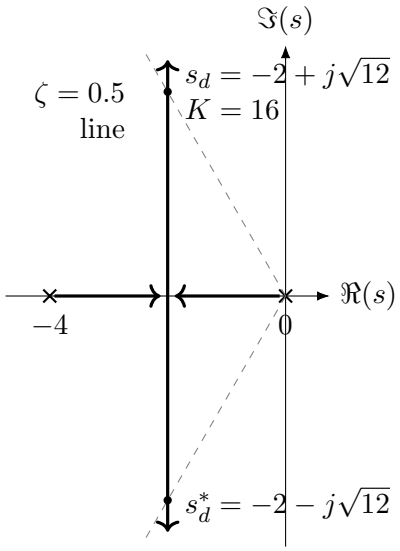


Figure 3: The $\zeta = 0.5$ lines intersect the root locus at the desired closed-loop poles.

The ideal second-order percentage overshoot is $100e^{-\pi/\sqrt{3}} \approx 16.3\%$, and the 2% settling-time estimate is $T_s \approx 2$ s. The real part remains -2 for all $K > 4$, so gain alone cannot arbitrarily decrease the settling time.

Question for Discussion

If we also require $T_s < 1$ s, can changing K alone satisfy it for this plant? What type of change to the controller might be needed?

7 Main Takeaways

1. Root locus tracks *closed-loop pole locations* as the nonnegative gain varies.
2. The angle condition finds admissible points; the magnitude condition gives their gains.
3. Poles, zeros, real-axis segments, asymptotes, and breakaway points determine the sketch.
4. Routh–Hurwitz and imaginary-axis crossing calculations identify stable gain ranges.
5. Desired transient performance can be translated into desired pole locations; second-order formulas are not universally exact for higher-order plants.

Homework

Problem 1. For $G(s) = 1/[s(s+6)]$, determine the root-locus real-axis segments and breakaway point. Find the poles at $K = 9$ and $K = 25$.

Problem 2. For $G(s) = (s+1)/[s(s+2)(s+5)]$, find the number of branches, real-axis segments, asymptote angles, and centroid. Sketch the root locus using these landmarks.

Problem 3. For $G(s) = 1/[s(s+2)(s+4)]$, derive the positive-gain stability interval using Routh–Hurwitz and find the imaginary-axis crossing frequency and corresponding gain.

Problem 4. For $G(s) = 1/[s(s+4)]$, choose K to achieve damping ratio $\zeta = 0.8$; determine the closed-loop poles and estimate percentage overshoot.

Problem 5. For $G(s) = 1/[s(s+2)]$, use the angle condition to determine whether $s_d = -1 + j\sqrt{3}$ lies on the positive-gain root locus. If it does, use the magnitude condition to find K and verify the resulting closed-loop poles.

Lecture 8 Homework Solutions

Problem 1. For $G(s) = 1/[s(s+6)]$, determine the root-locus real-axis segments and breakaway point. Find the poles at $K = 9$ and $K = 25$.

Solution 1. The open-loop poles are 0 and -6, with no zeros. Only $(-6, 0)$ is a real-axis segment. Since

$$K(s) = -s(s+6), \qquad \frac{dK}{ds} = -2s - 6,$$

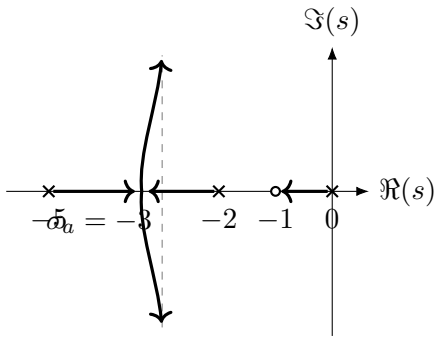
the breakaway point is $\boxed{s_b = -3}$, where $K_b = 9$. The characteristic polynomial is $s^2 + 6s + K$. Thus $K = 9$ gives a double pole at -3, while $K = 25$ gives $\boxed{s = -3 \pm 4j}$.

Problem 2. For $G(s) = (s+1)/[s(s+2)(s+5)]$, find the number of branches, real-axis segments, asymptote angles, and centroid. Sketch the root locus using these landmarks.

Solution 2. The poles are $0, -2, -5$ and the zero is -1. There are 3 branches. Real-axis segments (odd count to the right) are $\boxed{(-5, -2) \text{ and } (-1, 0)}$. There are $n - m = 2$ asymptotes, at $\boxed{90^\circ, 270^\circ}$, with centroid

$$\boxed{\sigma_a = \frac{(0 - 2 - 5) - (-1)}{2} = -3.}$$

One branch moves from the pole at 0 to the zero at -1. The branches from -5 and -2 meet on $(-5, -2)$, leave the real axis as a conjugate pair, and approach the vertical asymptotes through $\Re(s) = -3$.



Problem 3. For $G(s) = 1/[s(s+2)(s+4)]$, derive the positive-gain stability interval using Routh–Hurwitz and find the imaginary-axis crossing frequency and corresponding gain.

Solution 3. The characteristic polynomial is $s^3 + 6s^2 + 8s + K$. Its Routh first column is $1, 6, (48-K)/6, K$; hence $\boxed{0 < K < 48}$. At $K = 48$ it factors as $(s+6)(s^2+8)$, giving $\boxed{\omega = \sqrt{8} \text{ rad/s}}$.

Problem 4. For $G(s) = 1/[s(s+4)]$, choose K to achieve damping ratio $\zeta = 0.8$; determine the closed-loop poles and estimate percentage overshoot.

Solution 4. Since $\zeta = 2/\sqrt{K}$, choosing $\zeta = 0.8$ yields $\boxed{K = 6.25}$. The poles are $-2 \pm 1.5j$. The standard second-order formula gives

$$\boxed{M_p = 100e^{-\pi(0.8)/0.6}\% \approx 1.52\%.}$$

Problem 5. For $G(s) = 1/[s(s+2)]$, use the angle condition to determine whether $s_d = -1 + j\sqrt{3}$ lies on the positive-gain root locus. If it does, use the magnitude condition to find K and verify the resulting closed-loop poles.

Solution 5. At $s_d = -1 + j\sqrt{3}$, the vectors from the poles at 0 and -2 have angles 120° and 60°. Therefore

$$\angle G(s_d) = -(120^\circ + 60^\circ) = -180^\circ,$$

so s_d lies on the positive-gain root locus. Moreover,

$$K = \frac{1}{|G(s_d)|} = |s_d|\,|s_d + 2| = 2 \cdot 2 = \boxed{4}.$$

The characteristic polynomial is $s^2 + 2s + 4$, whose roots are $\boxed{-1 \pm j\sqrt{3}}$.