ME 41100: System Dynamics and Control
Lecture 7: Proportional–Integral–Derivative (PID) Control

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Learning Objectives. By the end of this lecture, you should be able to:

- explain the roles and limitations of proportional, integral, and derivative action;
- derive closed-loop equations for P, PI, PD, and PID control;
- use transient, steady-state, and stability requirements to select controller gains for simple plants;
- recognize derivative kick, noise amplification, actuator saturation, and integral windup.

1 From Analysis to Controller Design

$$\text{Physical System} \longrightarrow \text{Dynamic Model} \longrightarrow \text{Analyze} \longrightarrow \boxed{\text{Design}} \longrightarrow \text{Verify}.$$

Lecture 5 connected closed-loop poles with transient response. Lecture 6 taught us to test stability *before* computing steady-state errors. We now ask how to choose a controller that deliberately changes both.

Design question. How can feedback reduce tracking error without making the closed-loop response too oscillatory or unstable?

Notation and assumptions. For unity negative feedback, let $P(s)$ be the plant, $C(s)$ the controller, and $e(t) = r(t) - y(t)$ the tracking error. With zero initial conditions,

$$L(s) = C(s)P(s), \qquad T(s) = \frac{L(s)}{1 + L(s)}, \qquad S(s) = \frac{1}{1 + L(s)}.$$

All steady-state conclusions below require a stable closed loop. Derivatives in the ideal controller are mathematical idealizations; practical implementation requires filtering.

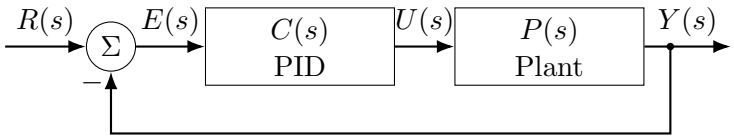


Figure 1: Unity negative feedback with a dynamic controller.

2 The Three Basic Control Actions

The *parallel-form* ideal PID law is

$$\boxed{u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,\mathrm{d}\tau + K_d \dot{e}(t),} \tag{1}$$

where an arbitrary initial integral state may also be specified. With zero initial integral state and zero initial signals,

$$\boxed{C(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}.}$$

The physical units of K_p, K_i, and K_d depend on the units of the error and control input. The three actions use different information about error.

Term	What it reacts to	Common benefit and limitation
$K_p e$	Present error	Stronger correction; may increase oscillation and cannot generally eliminate offset
$K_i \int e$	Accumulated past error	Removes constant offset if stable; may reduce stability margins and wind up
$K_d \dot{e}$	Rate of error change	Often adds damping; sensitive to measurement noise and reference jumps

These are *qualitative tendencies*, not universal rules for arbitrary plants.

A useful first summary.

- **P** Strengthens the immediate correction and changes response speed.
- **I** Improves steady-state accuracy by accumulating persistent error.
- **D** Adds anticipatory action and often improves damping and transient response.

These are common tendencies, not universal guarantees; the actual effects depend on the plant and closed-loop stability.

2.1 Proportional Control: Respond to the Present Error

$$\boxed{u(t) = K_p e(t), \qquad C_P(s) = K_p.}$$

For the stable plant $P(s) = 1/(s+2)$ and a unity-feedback step reference,

$$T_P(s) = \frac{K_p}{s + 2 + K_p}, \qquad e_{\text{ss}} = \frac{2}{2 + K_p} \quad (K_p > -2).$$

Increasing $K_p > 0$ makes the pole more negative and reduces the offset *in this example*; it does not yield zero offset at finite gain.

Question for Discussion

If a larger K_p reduces steady-state error, is it always safe to increase it without limit? Recall Lecture 6's example $P(s) = 1/(s+2)^3$.

2.2 Integral Control: Respond to Accumulated Error

Define an integral state $z(t)$:

$$\dot{z}(t) = e(t), \qquad u_I(t) = K_i z(t).$$

A persistent positive error causes z to increase. If the closed loop settles to an equilibrium with finite z and $K_i \neq 0$, then $\dot{z} = e = 0$: this is the basis of *zero constant tracking error*. Equilibrium existence alone does not prove convergence.

$$\boxed{C_{PI}(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}.}$$

For the example $P(s) = 1/(s+2)$, the characteristic polynomial becomes

$$\Delta_{PI}(s) = s(s+2) + K_p s + K_i = s^2 + (2 + K_p)s + K_i.$$

It is strictly Hurwitz if and only if $2 + K_p > 0$ and $K_i > 0$. Under these conditions,

$$\frac{E(s)}{R(s)} = \frac{s(s+2)}{s^2 + (2+K_p)s + K_i}, \qquad e_{\text{ss}} = \lim_{s\to 0} s\frac{E(s)}{s} = 0$$

for a unit step. *Integral control can eliminate offset, but only with a stable loop.*

2.3 Derivative Control: Respond to the Rate of Change

$$\boxed{u_D(t) = K_d \dot{e}(t), \qquad C_D(s) = K_d s.}$$

For a mechanical position variable, derivative feedback often corresponds to velocity feedback, which can increase effective damping. In contrast, differentiation of measured noise amplifies high-frequency noise.

Derivative kick. Because $\dot{e} = \dot{r} - \dot{y}$, an ideal step reference contains an impulse in $\dot{r}$. A standard practical alternative is to apply derivative action to the measured output:

$$\boxed{u(t) = K_p(r - y) + K_i z - K_d \dot{y}, \qquad \dot{z} = r - y.}$$

When the reference is constant, $-K_d\dot{y} = K_d\dot{e}$. At a reference discontinuity, however, the two laws command different control inputs: derivative-on-measurement avoids the setpoint derivative kick. It is therefore a two-input controller rather than a single block acting only on e.

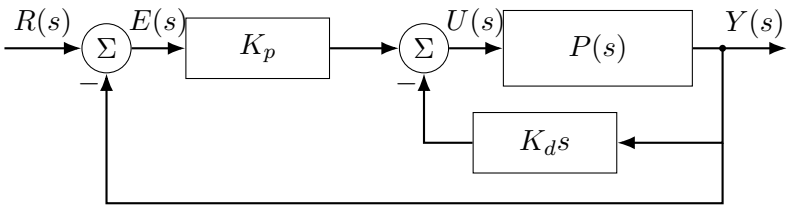


Figure 2: PD control with derivative action applied to the measured output.

3 A Mechanical Example: P versus PD

Consider the mass–spring–damper system already used in Lecture 6:

$$m\ddot{q} + b\dot{q} + kq = u, \qquad P(s) = \frac{Q(s)}{U(s)} = \frac{1}{ms^2 + bs + k}.$$

Take $m = 2$, $b = 3$, and $k = 4$ and let $r(t)$ be the desired position.

Example: One Mechanical Plant, Three Different Roles For the PID law with derivative on measurement,

$$u = K_p(r - q) + K_i z - K_d\dot{q}, \qquad \dot{z} = r - q,$$

the closed-loop equations are

$$\boxed{m\ddot{q} + (b + K_d)\dot{q} + (k + K_p)q = K_p r + K_i z, \qquad \dot{z} = r - q.}$$

This equation makes the three roles visible:

- K_p increases the immediate restoring action and generally reduces position error;
- K_d increases the effective damping from b to $b + K_d$, improving the transient response without changing a constant equilibrium;
- if the PID closed loop is stable and reaches an equilibrium, $\dot{z} = r - q = 0$, so integral action enforces zero constant tracking error.

Example: Why P Control Cannot Meet Both Specifications in This Example Under $u = K_p(r - q)$, the closed-loop equation is

$$2\ddot{q} + 3\dot{q} + (4 + K_p)q = K_p r.$$

Thus, for $K_p > 0$,

$$\omega_n = \sqrt{\frac{4 + K_p}{2}}, \qquad \zeta = \frac{3}{2\sqrt{2(4 + K_p)}}, \qquad e_{\text{ss}} = \frac{4}{4 + K_p}$$

for a unit step. Requiring $e_{\text{ss}} \le 20\%$ gives $K_p \ge 16$. At $K_p = 16$,

$$\omega_n = \sqrt{10}, \quad \zeta = \frac{3}{4\sqrt{10}} \approx 0.237, \quad \text{PO} \approx 46.4\%.$$

A stronger proportional gain reduces offset, but the resulting oscillation violates a 20% overshoot limit.

Example: Add Derivative Action to Improve Damping Use derivative on measurement (for constant references this agrees with ideal PD error feedback):

$$u = K_p(r - q) - K_d\dot{q}.$$

Then

$$\boxed{2\ddot{q} + (3 + K_d)\dot{q} + (4 + K_p)q = K_p r.}$$

The closed-loop characteristic polynomial is $2s^2 + (3 + K_d)s + (4 + K_p)$ and

$$\boxed{\omega_n = \sqrt{\frac{4 + K_p}{2}}, \quad \zeta = \frac{3 + K_d}{2\sqrt{2(4 + K_p)}}.}$$

Choose $K_p = 16$ to meet the 20% final-error requirement. The overshoot limit $\text{PO} \le 20\%$ requires

$$\zeta \ge \frac{-\ln(0.20)}{\sqrt{\pi^2 + \ln^2(0.20)}} \approx 0.456.$$

We select $\zeta = 0.7$ to provide damping margin and then solve for K_d:

$$K_d = 2\zeta\sqrt{2(4+K_p)} - 3 = 2(0.7)\sqrt{40} - 3 \approx 5.85.$$

Consequently $\omega_n = \sqrt{10}$, $\zeta = 0.7$, and the standard second-order estimate gives

$$\mathrm{PO} = 100\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \approx 4.60\%, \qquad t_s^{2\%} \approx \frac{4}{\zeta\omega_n} \approx 1.81 \text{ s}.$$

This example's final value is $q_\infty = 16/(4+16) = 0.8$; derivative feedback changes transient behavior, not its steady-state position offset.

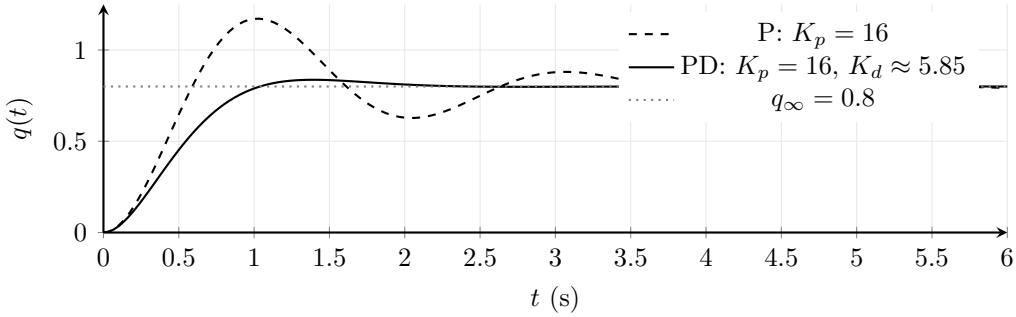


Figure 3: Derivative feedback improves the transient response without changing the final value in this example.

P versus PD in this plant. K_p determines the effective stiffness and the step offset; K_d changes effective damping. This convenient separation is particular to this model, not a universal PID tuning rule.

4 PI and PID: Zero Offset versus Stability

Return to the same mechanical plant, and consider the ideal controller

$$C(s) = K_p + \frac{K_i}{s} + K_d s.$$

The modeled characteristic equation is

$$\begin{aligned} 1 + C(s)P(s) &= 0, \\ \Delta(s) &= s(2s^2+3s+4) + K_d s^2 + K_p s + K_i \\ &= \boxed{2s^3 + (3+K_d)s^2 + (4+K_p)s + K_i.} \end{aligned}$$

Controller	Characteristic polynomial
P	$2s^2 + 3s + (4+K_p)$
PD	$2s^2 + (3+K_d)s + (4+K_p)$
PI	$2s^3 + 3s^2 + (4+K_p)s + K_i$
PID	$2s^3 + (3+K_d)s^2 + (4+K_p)s + K_i$

Stability test with Routh–Hurwitz. The Routh table for the ideal PID characteristic polynomial is

$$\begin{array}{c|cc} s^3 & 2 & 4+K_p \\ s^2 & 3+K_d & K_i \\ s^1 & \dfrac{(3+K_d)(4+K_p)-2K_i}{3+K_d} & 0 \\ s^0 & K_i & \end{array}.$$

Hence strict asymptotic stability holds exactly when

$$\boxed{3+K_d>0, \quad K_i>0, \quad (3+K_d)(4+K_p)>2K_i.}$$

These inequalities imply $4+K_p>0$. The condition presumes the ideal controller and the modeled plant, with no hidden unstable modes or problematic cancellations.

Example: An Integral Gain That Is Too Large Choose $K_p=8$, $K_d=0$. The characteristic polynomial becomes

$$2s^3+3s^2+12s+K_i.$$

The Routh condition is $3(12)>2K_i$, so the closed loop is stable precisely for

$$\boxed{0<K_i<18.}$$

At $K_i=4$, a unit step reference has zero steady-state tracking error. At $K_i=20$ the loop is unstable, so it is *incorrect* to claim zero long-time error simply because the controller contains an integrator.

Why the integral action removes constant error. With $K_i>0$ and a stable PID loop,

$$S(s)=\frac{s(2s^2+3s+4)}{2s^3+(3+K_d)s^2+(4+K_p)s+K_i}.$$

For a unit step $R(s)=1/s$, the Final Value Theorem gives

$$e_{\text{ss}}=\lim_{s\to 0} S(s)=0.$$

A constant disturbance $d(t)$ entering additively at the plant input gives

$$\frac{Y(s)}{D(s)}=\frac{P(s)}{1+C(s)P(s)}=\frac{s}{2s^3+(3+K_d)s^2+(4+K_p)s+K_i}.$$

Thus a stable closed loop also rejects a constant input disturbance in steady state. This is a concrete reason for using integral action in real mechanical systems.

5 Practical PID Implementation: A Brief Overview

Derivative filtering. An ideal differentiator amplifies high-frequency measurement noise. A realizable approximation is

$$\boxed{C_D(s)=K_d\frac{s}{1+\tau_f s}, \qquad \tau_f>0.}$$

The filtered controller has a different characteristic equation, so stability must be rechecked.

Actuator saturation and integral windup. Physical actuators satisfy $u_{\min} \le u \le u_{\max}$. During saturation, the integral state can continue growing even though the requested input cannot be delivered; the result is *integral windup*. A basic remedy is to pause integration when it would drive the actuator farther into saturation. Back-calculation is another common anti-windup method.

Optional after-class MATLAB verification (Control System Toolbox). For the mechanical example, compare a P controller with derivative-on-measurement PD for a constant-position step. Both have the reference numerator K_p:

```
s = tf('s');
Kp = 16; Kd = 2*0.7*sqrt(40)-3;
T_P = Kp/(2*s^2 + 3*s + 4 + Kp);
T_PD = Kp/(2*s^2 + (3+Kd)*s + 4 + Kp);
step(T_P,T_PD); grid on;
legend('P','PD with derivative on measurement');
stepinfo(T_P), stepinfo(T_PD)
```

For an ideal PID illustration (derivative applied to the error), one may use `C = pid(Kp,Ki,Kd)` and `T = feedback(C*P,1)` with $P = 1/(2s^2 + 3s + 4)$. In practice, specify derivative filtering and account for actuator saturation.

6 Main Takeaways

1. **P** responds to present error and often reduces offset, but increased gain can worsen oscillation or stability.
2. **I** accumulates error and can remove constant tracking error and constant input disturbances *if the closed loop remains stable.*
3. **D** responds to changes in error and can improve damping; derivative filtering and avoidance of setpoint kick are important.
4. The controller modifies the *characteristic polynomial.* Test stability before claiming performance.
5. Real-world PID design must account for sensor noise, actuator saturation, and integral windup.

Next lecture. Root locus: How do closed-loop poles move as a controller gain changes, and how can we use those trajectories for systematic design?

Homework

Problem 1. For $P(s) = 1/(s+3)$ and unity negative feedback, derive the closed-loop transfer function and unit-step steady-state error under $C(s) = K_p$. Determine the smallest $K_p \geq 0$ giving an error no greater than 10%.

Problem 2. For $P(s) = 1/(s+3)$ with $C(s) = K_p + K_i/s$, derive the characteristic polynomial, find necessary and sufficient strict stability conditions, and determine the unit-step steady-state error when stable. Then show that a constant disturbance applied at the plant input has zero steady-state effect on the output.

Problem 3. For $P(s) = 1/(2s^2 + 3s + 4)$, choose a PD controller with derivative on measurement to achieve step error at most 20% and damping ratio $\zeta = 0.6$. Use the smallest admissible K_p and calculate K_d.

Problem 4. For the same plant with ideal PID $C(s) = K_p + K_i/s + K_d s$, set $K_p = 8$ and $K_d = 1$. Determine the interval of K_i giving strict stability; test $K_i = 12$ and $K_i = 30$.

Problem 5. Explain (a) why ideal derivative action on a step reference causes a derivative kick and (b) how actuator saturation can cause integral windup. Describe one remedy for each.