ME 41100: System Dynamics and Control
Lecture 6: Stability and Steady-State Accuracy

Bo Wang
Department of Mechanical Engineering, The City College of New York

Learning Objectives. By the end of this lecture, you should be able to:

- determine closed-loop stability from pole locations;
- form the closed-loop characteristic equation and use the Routh array to determine stable parameter ranges;
- apply the Final Value Theorem only when its pole conditions are satisfied;
- derive steady-state tracking error for unity-feedback systems;
- use system type and static error constants to evaluate step, ramp, and parabolic tracking;
- explain the tradeoff between proportional gain, transient response, and steady-state accuracy.

1 Where We Are in the Course

$$\text{Physical System} \longrightarrow \text{Dynamic Model} \longrightarrow \text{Represent} \longrightarrow \boxed{\text{Analyze}} \longrightarrow \text{Design} \longrightarrow \text{Verify.}$$

Is the closed loop stable, and if it is, how accurately does it track the reference at steady state?

In this lecture, we finish the main analysis tools needed before controller design. The order matters:

$$\boxed{\text{closed-loop poles} \;\longrightarrow\; \text{stability} \;\longrightarrow\; \text{steady-state accuracy.}}$$

If a closed loop is unstable, a finite algebraic "steady-state error" has no physical meaning because the response does not settle. Stability therefore comes first.

Notation and Assumptions. We use $P(s)$ for the plant and $C(s)$ for the controller. For unity negative feedback,

$$L(s) = C(s)P(s), \qquad T(s) = \frac{Y(s)}{R(s)} = \frac{L(s)}{1+L(s)}, \qquad e(t) = r(t) - y(t).$$

Unless stated otherwise, we assume zero initial conditions when using transfer functions.

2 Stability and Pole Locations

Lecture 5 showed how a pole produces a natural mode. Recall the basic correspondence:

$$\frac{A}{s-p} \quad \longleftrightarrow \quad Ae^{pt}.$$

If $p = \sigma + j\omega$, then the magnitude of the mode contains the factor $e^{\sigma t}$. Therefore the real part of a pole determines whether that mode decays, persists, or grows.

Pole location	Typical mode	Long-time behavior
$\mathrm{Re}(p) < 0$	e^{pt}	Decays to zero
$\mathrm{Re}(p) > 0$	e^{pt}	Grows without bound
$p = \pm j\omega$	$\sin \omega t, \cos \omega t$	Sustained oscillation
$p = 0$	constant	Does not decay
repeated pole on the imaginary axis	$t^k e^{j\omega t}$	May grow with time

For the closed-loop systems considered in this lecture, we will use the following working criterion.

Closed-loop stability criterion. The closed loop is *asymptotically stable* if all closed-loop poles lie strictly in the open left half-plane:

$$\boxed{\mathrm{Re}(p_i) < 0 \quad \text{for every closed-loop pole.}}$$

A pole in the right half-plane makes the response *unstable*. A pole on the imaginary axis means the response does not asymptotically decay.

This criterion is a direct continuation of Lecture 5: stability is determined by the long-time behavior of the natural modes associated with the closed-loop poles.

Example: Reading Stability Directly from Poles Consider three characteristic polynomials:

$$\Delta_1(s) = (s+1)(s+4), \qquad \Delta_2(s) = (s-1)(s+4), \qquad \Delta_3(s) = s^2 + 4.$$

The first has poles $-1, -4$ and is stable. The second has a pole at $+1$ and is unstable. The third has poles $\pm j2$ and produces sustained oscillation, so it is not asymptotically stable.

3 Closed-Loop Characteristic Equations

Return to unity negative feedback:

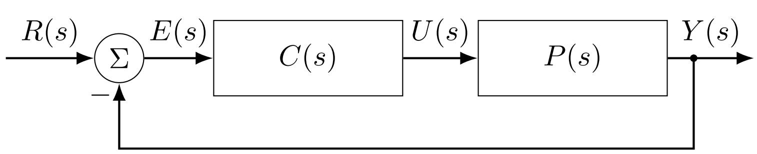

Figure 1: Unity negative feedback.

If

$$P(s) = \frac{N_P(s)}{D_P(s)}, \qquad C(s) = \frac{N_C(s)}{D_C(s)},$$

then the CLTF

$$T(s) = \frac{N_C(s)N_P(s)}{D_C(s)D_P(s) + N_C(s)N_P(s)}.$$

Hence the closed-loop characteristic polynomial is

$$\boxed{\Delta(s) = D_C(s)D_P(s) + N_C(s)N_P(s).}$$

The roots of $\Delta(s)$ are the closed-loop poles.

Example: Feedback Can Stabilize an Unstable Plant For $P(s) = 1/(s-1)$ and $C(s) = K$,

$$T(s) = \frac{K}{s-1+K}, \qquad p = 1 - K.$$

The closed loop is stable if and only if $K > 1$. Feedback changes the pole location; the open-loop plant pole is not automatically a closed-loop pole.

Example: Negative Feedback Does Not Guarantee Stability for Every Gain For $P(s) = 1/(s+2)^3$ and $C(s) = K > 0$,

$$\Delta(s) = (s+2)^3 + K = s^3 + 6s^2 + 12s + (8+K).$$

We will shortly find that the closed loop is stable only for $0 < K < 64$. Thus increasing the positive controller gain K in a negative-feedback loop can eventually destabilize a higher-order closed loop.

4 The Routh–Hurwitz Criterion

Why Use a Routh Array? For first- and second-order polynomials, roots are easy to calculate directly. For higher-order polynomials, especially when a coefficient contains an adjustable gain, solving explicitly for every root is inconvenient. The Routh–Hurwitz criterion determines how many roots lie in the right half-plane without solving for all of them.

Start with a Cubic For

$$\Delta(s) = s^3 + a_2 s^2 + a_1 s + a_0,$$

place alternating coefficients in the first two rows, then compute the remaining entries:

$$\begin{array}{c|cc} s^3 & 1 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & (a_2 a_1 - a_0)/a_2 & 0 \\ s^0 & a_0 & \end{array}.$$

The stability information is contained in the first column.

For the ordinary case, the number of sign changes in the first column equals the number of right-half-plane roots. With a positive leading coefficient, strict stability therefore requires every first-column entry to be positive.

Therefore strict stability is equivalent to

$$\boxed{a_2 > 0, \qquad a_1 > 0, \qquad a_0 > 0, \qquad a_2 a_1 > a_0.}$$

Positive coefficients alone are not sufficient for a cubic or higher-order polynomial.

Example: Finding a Stable Gain Interval Consider

$$L(s) = \frac{K}{s(s+1)(s+2)}.$$

The characteristic polynomial is

$$\Delta(s) = s(s+1)(s+2) + K = s^3 + 3s^2 + 2s + K.$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & (6-K)/3 & 0 \\ s^0 & K & \end{array}.$$

Strict positivity of the first column gives

$$\boxed{0 < K < 6.}$$

For $K = 8$, the first-column signs are $+, +, -, +$, so there are two right-half-plane roots and the closed loop is unstable. At the boundaries,

$$K = 0: \quad \Delta = s(s+1)(s+2), \qquad K = 6: \quad \Delta = (s+3)(s^2+2).$$

Neither endpoint is asymptotically stable. At $K = 6$, the poles include $\pm j\sqrt{2}$.

General Construction For

$$\Delta(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \qquad a_n > 0,$$

start with alternating coefficients and pad missing coefficients with zeros:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ s^0 & \cdots & & & \end{array}$$

The first entries of the next rows are

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \qquad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}},$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1}b_2}{b_1}.$$

Repeat the same pattern until the s^0 row is reached. If necessary, multiply the entire polynomial by -1 so that the leading coefficient is positive. A positive rescaling of any row is also allowed and can simplify arithmetic.

Example: A Fourth-Order Example Let

$$\Delta(s) = s^4 + 2s^3 + 3s^2 + 4s + 5.$$

The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & 1 & 5 & 0 \\ s^1 & -6 & 0 & 0 \\ s^0 & 5 & & \end{array}.$$

The first-column signs are $+, +, +, -, +$. There are two sign changes, so the polynomial has two right-half-plane roots and the system is unstable.

A Note on Special Cases*. Occasionally a first-column entry becomes zero, or an entire row becomes zero. These cases require a modified Routh procedure. We will not develop those procedures in this lecture. When a parameter reaches a suspected stability boundary, we will instead check that boundary directly by factoring the characteristic polynomial when possible.

Example: Returning to the Stable Plant with Excessive Gain For

$$\Delta(s) = s^3 + 6s^2 + 12s + (8 + K),$$

the cubic conditions give

$$8 + K > 0, \qquad 6 \cdot 12 > 8 + K.$$

For the intended $K > 0$,

$$\boxed{0 < K < 64.}$$

At $K = 64$,

$$\Delta = (s + 6)(s^2 + 12),$$

so a pair of poles lies on the imaginary axis. Increasing K beyond this value moves roots into the right half-plane.

5 Steady-State Response and the Final Value Theorem

Once a stable transient has decayed, the final value of a signal can often be found without computing the complete inverse Laplace transform. For a rational transform $F(s)$ of $f(t)$, the Final Value Theorem states

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

provided all poles of $sF(s)$ lie strictly in the open left half-plane after algebraic simplification.

> **Do not use the Final Value Theorem mechanically.** First check that the signal can actually settle. A finite value of $\lim_{s\to 0} sF(s)$ does not by itself prove that $f(t)$ has a final value.

Example: A Valid Application For

$$G(s) = \frac{6}{(s+2)(s+3)}$$

with a unit-step input,

$$Y(s) = \frac{6}{s(s+2)(s+3)}, \qquad sY(s) = \frac{6}{(s+2)(s+3)}.$$

The poles of $sY(s)$ are -2 and -3, so

$$y_\infty = \lim_{s\to 0} sY(s) = \frac{6}{2\cdot 3} = 1.$$

Example: Two Invalid Applications For $G(s) = 1/(s-1)$ with a unit-step input,

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{1}{s-1} = -1,$$

but the actual response contains e^t and diverges. The right-half-plane pole invalidates the theorem. For $G(s) = 1/(s^2+1)$, the same formal calculation gives 1, but the unit-step response is $1-\cos t$, which never converges. The imaginary-axis poles again invalidate the theorem.

DC Gain and Constant Inputs For a stable proper transfer function with finite DC gain $G(0)$, a step of amplitude A gives

$$\boxed{y_\infty = AG(0).}$$

For the mechanical system

$$m\ddot{q} + b\dot{q} + kq = A,$$

a constant equilibrium satisfies $\dot{q} = \ddot{q} = 0$, so $q_{\mathrm{ss}} = A/k$. Here q_{ss} denotes the final steady-state value for a constant input. The same value follows from the DC gain $P(0) = 1/k$. Stability is what guarantees that the response actually approaches this equilibrium.

6 Steady-State Tracking Error

For unity negative feedback,

$$E(s) = R(s) - Y(s), \qquad Y(s) = L(s)E(s).$$

Therefore

$$E(s) = R(s) - L(s)E(s) \quad \Longrightarrow \quad \boxed{E(s) = \frac{R(s)}{1+L(s)}.}$$

If the closed loop is stable and the Final Value Theorem applies to the error,

$$\boxed{e_{\mathrm{ss}} = \lim_{s\to 0} \frac{sR(s)}{1+L(s)}.}$$

This is the basic equation. The system-type formulas below are consequences of it.

Step Tracking For a step of amplitude A, $R(s) = A/s$, hence

$$e_{\mathrm{ss}} = \lim_{s \to 0} \frac{A}{1 + L(s)}.$$

If $L(0)$ is finite,

$$\boxed{e_{\mathrm{ss}} = \frac{A}{1 + L(0)}.}$$

A larger positive low-frequency loop gain reduces the step error, provided the resulting closed loop remains stable.

Example: A Type-0 Loop Let

$$P(s) = \frac{1}{s+2}, \qquad C(s) = K > 0.$$

Then

$$T(s) = \frac{K}{s + 2 + K}.$$

The closed-loop pole is $-(2 + K)$, so the closed loop is stable. For a unit step,

$$y_\infty = \frac{K}{2+K}, \qquad e_{\mathrm{ss}} = 1 - y_\infty = \frac{2}{2+K}.$$

At $K = 8$, $y_\infty = 0.8$ and $e_{\mathrm{ss}} = 0.2$.

7 System Type and Static Error Constants

What "System Type" Counts For the unity-feedback loop transfer function $L(s)$, the *system type* is the number of uncanceled poles at the origin.

Loop transfer function	Type	Reason
$5/[(s+1)(s+2)]$	0	No pole at the origin
$5/[s(s+2)]$	1	One pole at the origin
$5(s+1)/[s^2(s+3)]$	2	Two poles at the origin

System type is not the order of the plant and is not the number of closed-loop poles.

Step, Ramp, and Parabolic References The standard unit reference signals are

Reference	$r(t),\ t \geq 0$	$R(s)$
Step	1	$1/s$
Ramp	t	$1/s^2$
Parabolic	$t^2/2$	$1/s^3$

.

The factor $1/2$ makes the second derivative of the parabolic reference equal to one.

Static Error Constants Define

$$\boxed{K_p = \lim_{s\to 0} L(s), \qquad K_v = \lim_{s\to 0} sL(s), \qquad K_a = \lim_{s\to 0} s^2 L(s).}$$

These are the standard position, velocity, and acceleration error constants; they are not controller gains.

For example, a unit ramp has $R(s) = 1/s^2$, so

$$e_{\text{ss}} = \lim_{s\to 0} \frac{s}{1+L(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s[1+L(s)]} = \frac{1}{K_v},$$

provided the closed loop is stable and $0 < K_v < \infty$. The parabolic result follows in the same way after multiplying by one additional factor of $1/s$.

For a stable unity-feedback loop,

$$\text{unit step:} \qquad e_{\text{ss}} = \frac{1}{1+K_p},$$

$$\text{unit ramp:} \qquad e_{\text{ss}} = \frac{1}{K_v} \quad \text{when } 0 < K_v < \infty,$$

$$\text{unit parabolic:} \qquad e_{\text{ss}} = \frac{1}{K_a} \quad \text{when } 0 < K_a < \infty.$$

The resulting system-type table is

System type	Unit step	Unit ramp	Unit parabolic
0	$1/(1+K_p)$	∞	∞
1	0	$1/K_v$	∞
2	0	0	$1/K_a$

Use the type table only after checking closed-loop stability. Adding a pole at the origin can improve steady-state tracking, but it also changes the closed-loop characteristic equation and may affect stability.

Example: A Stable Type-1 Loop Consider

$$L(s) = \frac{10}{s(s+2)}.$$

The characteristic polynomial is

$$s(s+2) + 10 = s^2 + 2s + 10,$$

with poles $-1 \pm j3$, so the closed loop is stable. The loop is Type 1, and

$$K_v = \lim_{s\to 0} \frac{10}{s+2} = 5.$$

Therefore the unit-step error is zero and the unit-ramp error is

$$\boxed{e_{\text{ss}} = \frac{1}{5} = 0.2.}$$

For a unit ramp, the same result follows directly from

$$E(s) = \frac{s+2}{s(s^2+2s+10)}, \qquad sE(s) = \frac{s+2}{s^2+2s+10}.$$

Example: Why Stability Must Be Checked First Consider

$$L(s) = \frac{10}{s^2}.$$

This loop has two poles at the origin and is therefore Type 2. However, the closed-loop characteristic equation is

$$s^2 + 10 = 0,$$

with poles $\pm j\sqrt{10}$. The closed loop is not asymptotically stable, so the usual stable Type-2 steady-state conclusions cannot be applied.

8 What Does Proportional Feedback Change?

Continue the Example from Lecture 4 The plant and proportional position controller are

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t), \qquad u(t) = K\big[r(t) - q(t)\big].$$

Substitution gives

$$\boxed{m\ddot{q}(t) + b\dot{q}(t) + (k+K)q(t) = Kr(t).}$$

Thus

$$T(s) = \frac{Q(s)}{R(s)} = \frac{K}{ms^2 + bs + k + K}, \qquad L(s) = \frac{K}{ms^2 + bs + k}.$$

Feedback increases the effective stiffness from k to $k + K$; it does not change m or b.

Stability and Transient Parameters For $m > 0$, the closed-loop quadratic is stable when

$$b > 0, \qquad k + K > 0.$$

With the usual positive physical parameters and $K > 0$, this condition holds. Matching the standard second-order form gives

$$\boxed{\omega_n = \sqrt{\frac{k+K}{m}}, \qquad \zeta = \frac{b}{2\sqrt{m(k+K)}}, \qquad K_0 = T(0) = \frac{K}{k+K}.}$$

In the underdamped case,

$$p_{1,2} = -\frac{b}{2m} \pm j\sqrt{\frac{k+K}{m} - \frac{b^2}{4m^2}}.$$

The real part is

$$\boxed{\sigma = \zeta\omega_n = \frac{b}{2m},}$$

which is independent of K.

Therefore increasing K increases the natural frequency and decreases the damping ratio, while leaving the exponential decay rate unchanged. Within the underdamped regime, the usual estimate

$$t_s^{2\%} \approx \frac{4}{\sigma} = \frac{8m}{b}$$

is also unchanged.

Steady-State Accuracy and Physical Equilibrium For a constant reference r_0,

$$q_{\text{ss}} = \frac{K}{k+K} r_0, \qquad \boxed{e_{\text{ss}} = r_0 - q_{\text{ss}} = \frac{k}{k+K} r_0.}$$

The same result follows from static force balance. At a nonzero equilibrium displacement the spring requires a nonzero force:

$$u_{\text{ss}} = K e_{\text{ss}} = k q_{\text{ss}}.$$

Thus proportional control generally needs a nonzero tracking error to generate the force required to hold the spring at a nonzero displacement.

Quantity	Effect of increasing K	Reason
Natural frequency ω_n	Increases	Effective stiffness becomes $k + K$
Damping ratio ζ	Decreases	b is fixed while ω_n increases
Decay rate σ	Unchanged	$\sigma = b/(2m)$
Step tracking error	Decreases	$e_{\text{ss}}/r_0 = k/(k + K)$
Normalized overshoot	Increases once underdamped	The damping ratio decreases
Initial control effort	Increases	$u(0^+) = Kr_0$ for zero initial displacement

Numerical Values from Lecture 4 Use $m = 2, \quad b = 3, \quad k = 4, \quad K = 8$. Then

$$T(s) = \frac{8}{2s^2 + 3s + 12} = \frac{4}{s^2 + 1.5s + 6},$$

$$\omega_n = \sqrt{6} \approx 2.449, \quad \zeta = \frac{3}{4\sqrt{6}} \approx 0.3062, \quad \sigma = 0.75, \quad \omega_d = \frac{\sqrt{87}}{4} \approx 2.332.$$

For a unit displacement reference,

$$q_{\text{ss}} = \frac{2}{3}, \qquad e_{\text{ss}} = \frac{1}{3},$$

and Lecture 5 formulas give

$$t_p = \frac{4\pi}{\sqrt{87}} \approx 1.347 \text{ s}, \qquad M_p = e^{-3\pi/\sqrt{87}} \approx 0.364,$$

$$t_s^{2\%} \approx 5.333 \text{ s}, \qquad q_{\max} = \frac{2}{3}(1 + M_p) \approx 0.909.$$

The response has 36.4% overshoot relative to its own final value 2/3, although the peak remains below the desired reference 1.

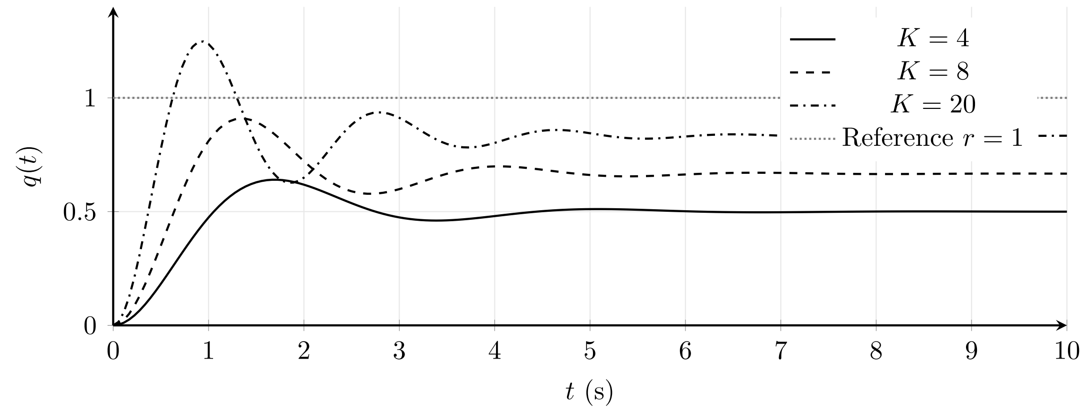

Figure 2: The same mechanical plant with three proportional gains. Greater gain improves final tracking but, once the response is underdamped, increases normalized overshoot.

A Simple Feasibility Check on Performance Requirements Suppose we require no more than 10% unit-step error and no more than 10% overshoot. The error requirement gives

$$\frac{4}{4+K} \leq 0.1 \quad \Longrightarrow \quad K \geq 36.$$

The overshoot requirement needs $\zeta \geq 0.591$. For this plant,

$$\zeta = \frac{3}{2\sqrt{2(4+K)}}.$$

Even as $K \to 0^+$, ζ approaches only $3/(2\sqrt{8}) \approx 0.530$, and it decreases as K increases. Therefore no positive proportional gain satisfies the 10% overshoot requirement, let alone both requirements together.

A single proportional gain cannot independently set steady-state accuracy, damping, and speed. This limitation motivates adding controller dynamics in Lecture 7.

9 Main Takeaways

1. Closed-loop stability is determined by the closed-loop poles. All poles must lie strictly in the left half-plane for asymptotic stability.
2. The closed-loop poles are roots of the characteristic polynomial obtained from the feedback interconnection.
3. The Routh–Hurwitz criterion finds right-half-plane roots and stable gain ranges without explicitly solving for every pole.
4. The Final Value Theorem may be used only when its pole condition is satisfied.
5. For a stable unity-feedback loop, system type and static error constants determine steady-state tracking errors for standard step, ramp, and parabolic references.
6. Increasing proportional gain can reduce steady-state error while simultaneously reducing damping and increasing overshoot or control effort.

Homework

Unless stated otherwise, assume zero initial conditions and unity negative feedback. Show the reasoning leading to each result. Use numerical software only as a check after the hand calculation.

Problem 1. (Pole locations, stability, and final values.) Treat each $G_i(s)$ below as the complete input-to-output transfer function. Determine whether its poles imply asymptotic stability. For a unit-step input, decide whether the Final Value Theorem is applicable and calculate the final value only when justified:

$$G_a(s) = \frac{2}{s+2}, \qquad G_b(s) = \frac{1}{s}, \qquad G_c(s) = \frac{4}{s^2+4}, \qquad G_d(s) = \frac{1}{s-2}.$$

Problem 2. (Routh gain interval.) For

$$L(s) = \frac{K}{s(s+2)(s+4)}, \qquad K > 0,$$

construct the Routh array and find the strict stability range. Determine the poles at the nonzero stability boundary. Count the right-half-plane roots when $K = 60$.

Problem 3. (Type and reference tracking.) For each loop below, first establish closed-loop stability, then determine system type and the steady-state errors for unit step, unit ramp, and unit parabolic references:

$$L_a(s) = \frac{4}{s+2}, \qquad L_b(s) = \frac{12}{s(s+3)}, \qquad L_c(s) = \frac{6(s+1)}{s^2(s+4)}.$$

Problem 4. (Direct use of the error transfer function.) Consider

$$L(s) = \frac{K}{s+3}, \qquad K > 0.$$

For a unit-step reference, derive $E(s)$ directly, determine e_{ss}, and find the smallest K that gives $e_{\text{ss}} \le 0.05$. Verify that the resulting closed loop is stable.

Problem 5. (The mechanical plant.) Use $m = 2$, $b = 3$, $k = 4$ and

$$u(t) = K[r(t) - q(t)].$$

Find the smallest $K > 0$ that gives at most 20% unit-step error. At that gain, find the poles, damping ratio, percent overshoot, and approximate 2% settling time. Does the response satisfy at most 20% overshoot as well?

Problem 6. (Can proportional gain meet both specifications?) For the same mechanical plant, suppose the requirements are

$$\frac{e_{\text{ss}}}{r_0} \le 0.10, \qquad \text{PO} \le 20\%.$$

Determine the range of $K > 0$ implied by each requirement. Is there a proportional gain that satisfies both? Explain what this result says about the limitations of proportional control.