ME 41100: System Dynamics and Control
Lecture 5: Time-Domain Response and Performance

Bo Wang

Department of Mechanical Engineering, The City College of New York

Learning Objectives. By the end of this lecture, you should be able to:

- distinguish zero-input, zero-state, transient, and steady-state components of a response;
- relate transfer-function poles to natural modes and derive first- and second-order step responses;
- calculate and interpret rise time, peak time, overshoot, and settling time;
- assess dominant-pole approximations and explain how zeros affect the response;

1 Where We Are in the Course

$$\text{Physical System} \longrightarrow \text{Dynamic Model} \longrightarrow \text{Represent} \longrightarrow \boxed{\text{Analyze}} \longrightarrow \text{Design} \longrightarrow \text{Verify.}$$

Given a transfer function, what does its time response look like, and which poles dominate the observed behavior?

Lecture 4 developed transfer functions for interconnected and feedback systems. We now begin the *analysis* step by connecting poles to natural modes and measurable time-domain performance:

$$\text{Transfer Function} \longrightarrow \text{Poles and Modes} \longrightarrow \text{Time Response} \longrightarrow \text{Performance Measures.}$$

Low-order models provide useful formulas, while dominant-pole approximations and zeros determine when those formulas remain reliable for higher-order systems.

Notation and Assumptions. Unless stated otherwise, transfer-function responses assume zero initial conditions. We write

$$G(s) = \frac{Y(s)}{U(s)}, \qquad T(s) = \frac{Y(s)}{R(s)}$$

for a general input–output map and a reference-to-output closed-loop map, respectively.

2 Time Response, Natural Modes, and Long-Term Behavior

Zero-Input and Zero-State Responses For a linear system, the complete response can be decomposed as

$$\boxed{y(t) = y_{\mathrm{zi}}(t) + y_{\mathrm{zs}}(t).}$$

The *zero-input response* is caused by initial stored energy, with the input set to zero. The *zero-state response* is caused by the input, with initial conditions set to zero. A transfer function directly describes the second of these:

$$Y_{\mathrm{zs}}(s) = G(s)U(s).$$

This does not mean that initial conditions are physically unimportant. It means we must account for them separately if they are nonzero.

Example: Consider $\dot{y}(t) + 2y(t) = u(t), \quad y(0) = y_0, \quad u(t) = 1 \quad (t \geq 0).$

Taking the Laplace transform gives

$$sY(s) - y_0 + 2Y(s) = \frac{1}{s},$$

so

$$Y(s) = \underbrace{\frac{y_0}{s+2}}_{\text{initial condition}} + \underbrace{\frac{1}{s(s+2)}}_{\text{input}}.$$

Since $1/[s(s+2)] = 1/(2s) - 1/[2(s+2)]$,

$$y(t) = \underbrace{y_0 e^{-2t}}_{y_{\mathrm{zi}}(t)} + \underbrace{\frac{1}{2}(1 - e^{-2t})}_{y_{\mathrm{zs}}(t)} = \frac{1}{2} + \left(y_0 - \frac{1}{2}\right) e^{-2t}.$$

Both initial conditions and the applied input can affect the coefficient of the natural mode e^{-2t}.

Transient and Steady-State Components For a stable system and an input having a well-defined long-term form, it is often useful to write

$$y(t) = y_{\mathrm{tr}}(t) + y_{\mathrm{ss}}(t), \qquad \lim_{t\to\infty} y_{\mathrm{tr}}(t) = 0.$$

In the preceding example, $y_{\mathrm{tr}} = (y_0 - 1/2)e^{-2t}$ and $y_{\mathrm{ss}} = 1/2$. This decomposition answers a different question from zero-input versus zero-state: it separates temporary behavior from long-term behavior.

- For a step input, a stable system typically approaches a constant.
- For a sinusoidal input, a stable system typically approaches a sinusoid at the same frequency; its steady-state response is not a constant.
- For a ramp input, the long-term output may grow with time even when the system is stable, because the input itself is unbounded.

An unstable mode does not become a decaying transient merely because it appears in the homogeneous solution. We must check its pole location.

How Poles Become Time-Domain Modes For a simple pole p, a partial-fraction term gives

$$\frac{A}{s-p} \quad \longleftrightarrow \quad Ae^{pt}.$$

Writing $p = \sigma + j\omega$ gives $e^{pt} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$. A conjugate pair therefore produces a real response of the form

$$e^{\sigma t}\big[A\cos(\omega t) + B\sin(\omega t)\big].$$

For a repeated pole, additional polynomial factors appear:

$$\frac{A}{(s-p)^q} \quad \longleftrightarrow \quad \frac{A}{(q-1)!} t^{q-1} e^{pt}.$$

The following table summarizes the basic building blocks.

Pole(s)	Typical natural mode	Behavior
$-a$, $a > 0$	e^{-at}	Exponential decay
$+a$, $a > 0$	e^{at}	Exponential growth
$-\sigma \pm j\omega$, $\sigma > 0$	$e^{-\sigma t}\cos\omega t$, $e^{-\sigma t}\sin\omega t$	Decaying oscillation
$\pm j\omega$, $\omega > 0$	$\cos\omega t$, $\sin\omega t$	Sustained oscillation
0	1	Constant natural mode
Repeated 0	$1, t, \ldots$	Possible polynomial growth

Poles determine the available natural modes; coefficients determine how strongly those modes appear. The coefficients depend on the input, initial conditions, and numerator. A mode may be absent from one particular measured response.

Example: Reading a Response from Its Partial Fractions Let

$$G(s) = \frac{3}{(s+1)(s+3)}, \qquad U(s) = \frac{1}{s}.$$

Then

$$Y(s) = \frac{3}{s(s+1)(s+3)} = \frac{1}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3},$$

and

$$\boxed{y(t) = 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.}$$

The constant 1 comes from the step forcing. The two decaying terms are associated with the system poles -1 and -3. The e^{-3t} term decays more quickly; the e^{-t} term controls the late transient. As checks, $y(0) = 0$ and $y(\infty) = G(0) = 1$.

Question for Discussion

A response contains $4e^{-5t} + 0.2e^{-t}$. Which term is initially larger? Which term eventually dominates the transient? Why are these different questions?

3 First-Order Systems

Standard Form and Physical Meaning A stable first-order model is

$$\boxed{G(s) = \frac{K_0}{\tau s + 1}}, \qquad \tau > 0,$$

with differential equation $\tau\dot{y}(t) + y(t) = K_0 u(t)$. Here $K_0 = G(0)$ is the DC gain, τ is the time constant in seconds, and the pole is

$$\boxed{p = -\frac{1}{\tau}.}$$

For example, a translating mass subject to viscous friction satisfies

$$m\dot{v}(t) + bv(t) = f(t), \qquad \frac{V(s)}{F(s)} = \frac{1/b}{(m/b)s + 1}.$$

Thus $K_0 = 1/b$ and $\tau = m/b$. A larger mass increases the time constant; greater viscous resistance reduces it but also reduces the final speed for the same constant force.

Derivation of the Step Response For a step of amplitude A, $U(s) = A/s$. With zero initial conditions,

$$Y(s) = \frac{AK_0}{s(\tau s + 1)} = AK_0 \left(\frac{1}{s} - \frac{1}{s + 1/\tau} \right).$$

Therefore,

$$\boxed{y(t) = AK_0(1 - e^{-t/\tau}), \qquad t \geq 0.}$$

The final value is AK_0. For a positive final value, the response approaches it monotonically and has no overshoot. At $t = \tau$,

$$\frac{y(\tau)}{y(\infty)} = 1 - e^{-1} \approx 0.632.$$

The time constant is the time required to complete about 63.2% of the total change, not the time required to finish the response.

With a nonzero initial condition, the solution is

$$y(t) = AK_0 + \big[y(0) - AK_0\big]e^{-t/\tau}.$$

The same time constant governs convergence from any initial output.

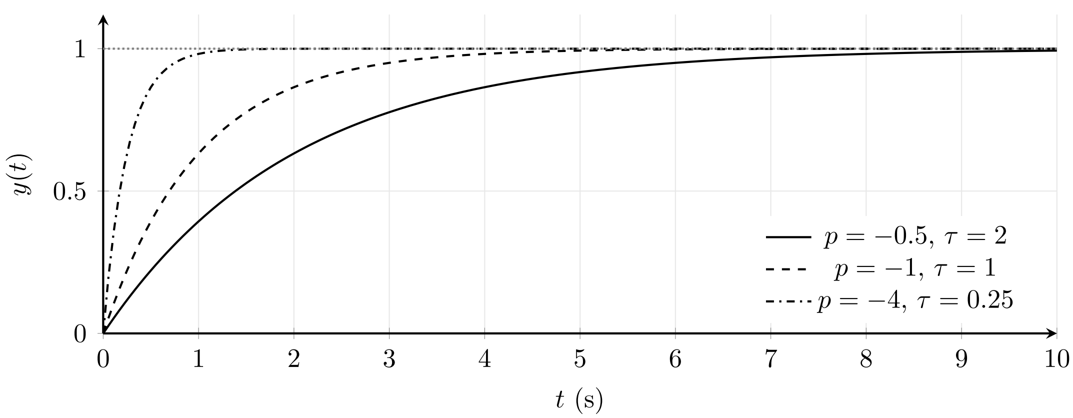

Figure 1: Unit-step responses with the same DC gain $K_0 = 1$. A first-order pole farther to the left gives a faster response.

Settling Time and Rise Time The settling time t_s is the earliest time after which the response stays inside a specified tolerance band around its final value. For a first-order step response, the normalized remaining difference is $e^{-t/\tau}$. For a tolerance fraction δ,

$$e^{-t_s/\tau} = \delta \quad \Longrightarrow \quad \boxed{t_s = -\tau \ln \delta.}$$

Thus

$$t_s = 3.912\tau \approx 4\tau \quad (2\%), \qquad t_s = 2.996\tau \approx 3\tau \quad (5\%).$$

The *10–90% rise time* is the time between first reaching 10% and first reaching 90% of the total change. Solving $1 - e^{-t/\tau} = q$ gives $t_q = -\tau \ln(1-q)$, so

$$\boxed{t_r^{10\text{–}90} = t_{0.9} - t_{0.1} = \tau \ln 9 \approx 2.197\tau.}$$

A first-order response never reaches its final value exactly in finite time. Its 0–100% rise time is therefore infinite; this is why the definition of rise time matters.

Example: Extracting Parameters from a Transfer Function For $G(s) = 6/(2s+3)$, first normalize the constant denominator coefficient:

$$G(s) = \frac{2}{(2/3)s + 1}, \qquad K_0 = 2, \qquad \tau = \frac{2}{3}\text{ s.}$$

For a unit step,

$$y(t) = 2(1 - e^{-1.5t}), \qquad y(\infty) = 2,$$
$$t_r^{10\text{–}90} \approx 1.465\text{ s}, \qquad t_s^{2\%} \approx 2.608\text{ s.}$$

Notice that the coefficient of s in an unnormalized denominator is not automatically the time constant.

4 Second-Order Systems and Pole Geometry

Standard Form The standard second-order model with no finite zeros is

$$\boxed{G(s) = K_0 \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.}$$

Here $\omega_n > 0$ is the undamped natural frequency in radians per second, ζ is the dimensionless damping ratio, and K_0 is the DC gain. Unless otherwise stated, we use $K_0 = 1$ to study the response shape independently of its amplitude. The poles solve the quadratic equation:

$$\boxed{p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.}$$

Connection to the Mass–Spring–Damper Model For

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t), \qquad m > 0,\ b > 0,\ k > 0,$$

the displacement-to-force transfer function is

$$P(s) = \frac{1}{ms^2 + bs + k} = \frac{1}{k}\frac{k/m}{s^2 + (b/m)s + k/m}.$$

Matching coefficients gives

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}, \qquad \zeta = \frac{b}{2\sqrt{mk}}, \qquad K_0 = \frac{1}{k}.}$$

The stiffness supplies a restoring force, mass resists acceleration, and damping dissipates energy. The ratio $b/(2\sqrt{mk})$ compares the actual damping with the critical damping value.

The Four Basic Damping Cases The pole pattern determines the qualitative response:

Damping	Poles	Unit-step behavior of the standard model
$\zeta > 1$	Two distinct negative real poles	Overdamped; monotone approach
$\zeta = 1$	Repeated pole at $-\omega_n$	Critically damped; monotone approach
$0 < \zeta < 1$	Complex conjugates in the LHP	Underdamped; decaying oscillation
$\zeta = 0$	$\pm j\omega_n$	Undamped; persistent oscillation

These descriptions apply to the standard no-zero model with positive gain and zero initial conditions. For $\zeta < 0$, the model has right-half-plane poles and is unstable.

For $\zeta > 1$, define positive rates

$$a = \omega_n(\zeta - \sqrt{\zeta^2 - 1}), \qquad b_1 = \omega_n(\zeta + \sqrt{\zeta^2 - 1}), \qquad b_1 > a.$$

The normalized unit-step response is

$$y(t) = 1 - \frac{b_1 e^{-at} - a e^{-b_1 t}}{b_1 - a}.$$

The slower rate a controls the late response. For critical damping, $\xi = 1$,

$$\boxed{y(t) = 1 - (1 + \omega_n t)e^{-\omega_n t}.}$$

Critical damping is the boundary between real and complex poles. For fixed ω_n, it gives the fastest monotone response in this standard family.

Underdamped Pole Geometry When $0 < \zeta < 1$, define

$$\sigma = \zeta\omega_n > 0, \qquad \omega_d = \omega_n\sqrt{1 - \zeta^2} > 0.$$

The poles are

$$\boxed{p_{1,2} = -\sigma \pm j\omega_d.}$$

Their distance from the origin is ω_n because $\sigma^2 + \omega_d^2 = \omega_n^2$. If θ is the angle measured from the negative real axis to the upper pole, then

$$\boxed{\zeta = \cos\theta, \qquad \omega_n = |p|, \qquad \sigma = -\operatorname{Re}(p), \qquad \omega_d = |\operatorname{Im}(p)|.}$$

The real part sets the exponential decay rate. The imaginary part sets the oscillation frequency. The pole angle determines the damping ratio, and the distance from the origin determines the natural frequency.

For example, poles $-2 \pm j3$ give $\sigma = 2$, $\omega_d = 3$, $\omega_n = \sqrt{13}$, and $\zeta = 2/\sqrt{13}$. Their natural oscillation envelope decays as e^{-2t}.

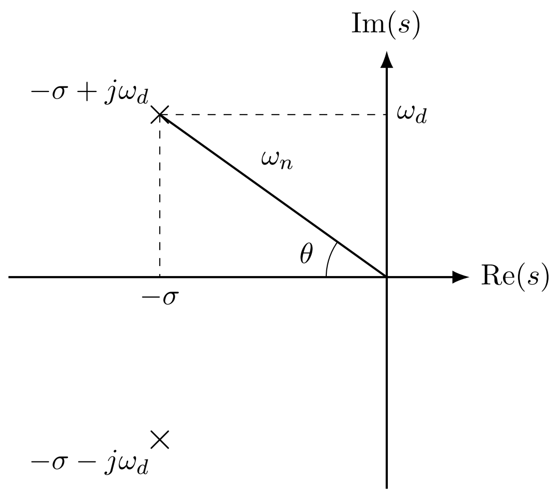

Figure 2: Geometry of an underdamped conjugate pole pair.

Derivation of the Underdamped Step Response For $K_0 = 1$ and a unit step,

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Separate the constant final-value contribution:

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \sigma}{(s+\sigma)^2 + \omega_d^2} - \frac{\sigma}{(s+\sigma)^2 + \omega_d^2}. \end{aligned}$$

Taking inverse transforms gives

$$\boxed{y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right].}$$

Equivalently, with $\phi = \cos^{-1}\zeta$,

$$\boxed{y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi), \quad 0 < \zeta < 1.}$$

For a step of amplitude A and DC gain K_0, multiply the normalized expression by AK_0. Check $y(0) = 0$, $\dot{y}(0) = 0$, and $y(\infty) = 1$. The zero initial slope is consistent with a displacement output: a finite force cannot instantaneously change the velocity of a mass initially at rest.

Question for Discussion

Two systems have poles $-1 \pm j2$ and $-1 \pm j5$. Which oscillates faster? Do their natural envelopes have different exponential decay rates? Must their percent overshoots be equal?

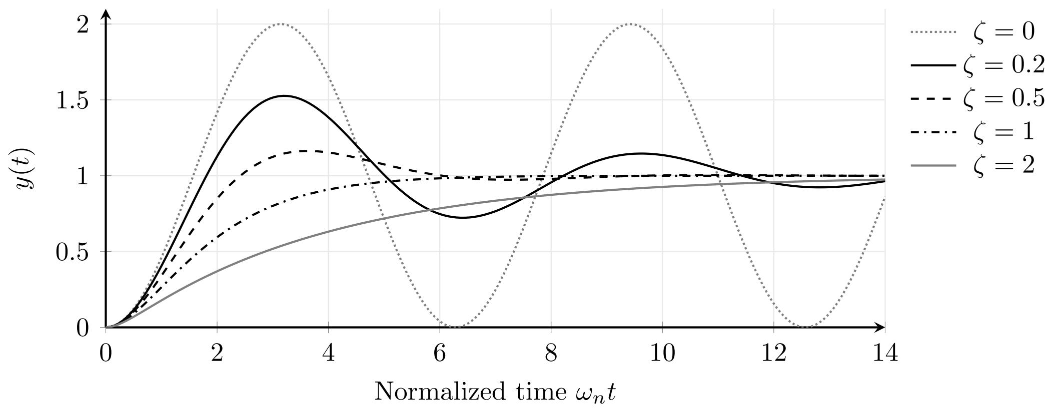

Figure 3: Damping changes the shape of the standard second-order unit-step response. All curves have the same ω_n and DC gain.

5 Time-Domain Performance Specifications

Define the Measurement Before Using a Formula Response metrics require explicit conventions and a clearly identified final value.

Unless otherwise stated, the formulas in this section apply to the zero-state unit-step response of a stable, no-zero, underdamped standard second-order model with a positive final value.

For a response starting from zero and approaching $y_\infty > 0$:

- **Rise time t_r:** time required for the response to rise between specified levels. We distinguish 10–90% rise time from first 0–100% rise time.
- **Peak time t_p:** time of the first maximum above the final value.
- **Overshoot ratio M_p:** peak excess divided by the final change:

$$M_p = \frac{y_{\max} - y_\infty}{y_\infty}, \qquad \mathrm{PO} = 100 M_p\%.$$

- **Settling time t_s:** earliest time after which $|y(t) - y_\infty| \leq \delta y_\infty$ for all subsequent times. Unless specified otherwise, $\delta = 0.02$.

For a nonzero initial output, normalize by the magnitude of the total change rather than blindly dividing by y_∞. Special conventions are needed if that change is zero.

Quantity	Formula	Interpretation
Peak time	$t_p = \pi/\omega_d$	Time of the first maximum
Overshoot ratio	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	Peak excess divided by the final change
2% settling time	$t_s \approx 4/(\zeta\omega_n)$	Common decay-rate estimate
First 0–100% rise time	$t_r = (\pi - \cos^{-1}\zeta)/\omega_d$	First crossing of the final value

A settling band is centered on the *final output*, not automatically on the desired reference. A system can settle within 2% of its own final output and still have a large tracking error.

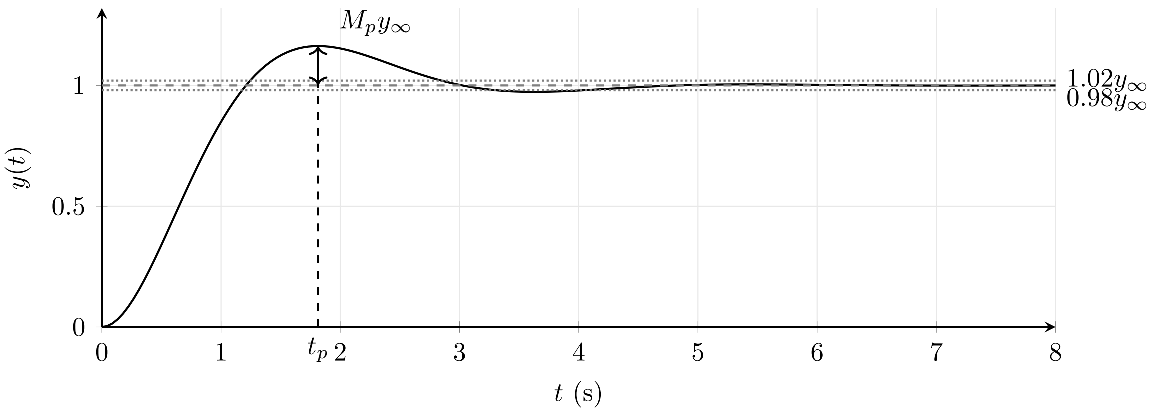

Figure 4: Performance measurements for $\zeta = 0.5$, $\omega_n = 2$, and $y_\infty = 1$. Settling requires staying inside the band, not merely entering it once.

Peak Time and Overshoot: A Short Derivation Differentiate the standard step response:

$$\dot{y}(t) = \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t).$$

The derivative is positive just after $t = 0$. Its first subsequent zero occurs at $\omega_d t = \pi$; this is the first maximum. Therefore,

$$\boxed{t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}.}$$

At that time, $\sin \pi = 0$ and $\cos \pi = -1$, giving

$$y(t_p) = 1 + e^{-\sigma\pi/\omega_d}.$$

Hence

$$\boxed{M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \qquad 0 < \zeta < 1.}$$

Overshoot depends only on ζ within this standard family. Increasing ω_n with ζ fixed compresses the time axis without changing the normalized peak height.

ζ	0.2	0.4	0.5	0.6	0.7
Percent overshoot	52.7%	25.4%	16.3%	9.5%	4.6%

Solving the overshoot equation for ζ gives

$$\boxed{\zeta = \frac{-\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}}, \qquad 0 < M_p < 1.}$$

Use the fraction $M_p = 0.10$ for 10% overshoot, not the number 10.

Settling Time: Approximation Versus Guaranteed Envelope The underdamped response satisfies

$$|y(t) - 1| \le \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}.$$

A sufficient time for the entire envelope to enter a tolerance band is

$$\boxed{t_{\mathrm{env}} = \frac{-\ln\left(\delta\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}.}$$

For this exact standard model, the actual settling time is no later than t_{env}. It may be earlier because the oscillatory factor need not touch the envelope at the last band crossing. The familiar estimates are

$$\boxed{t_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \quad (2\%), \qquad t_s \approx \frac{3}{\zeta\omega_n} \quad (5\%).}$$

These retain the dominant exponential decay and replace logarithmic factors by rounded constants. They are useful estimates for ordinary underdamped responses, not exact identities or universal upper bounds. In particular, do not apply $4/(\zeta\omega_n)$ to an overdamped system: its slow real pole can be much closer to the origin than $-\zeta\omega_n$.

Rise Time Conventions For the first 0–100% rise time of an underdamped standard system, set $y(t) = 1$ for the first time. The phase expression gives $\omega_d t + \phi = \pi$, so

$$\boxed{t_r^{0\text{–}100} = \frac{\pi - \cos^{-1}\zeta}{\omega_d}.}$$

For 10–90% rise time, solve the two crossing equations $y(t_{10}) = 0.1$ and $y(t_{90}) = 0.9$, then subtract. There is no comparably simple general closed form. Numerical computation is appropriate. Always state which convention is used when comparing hand calculations with software.

Example: A Complete Second-Order Calculation Consider

$$T(s) = \frac{25}{s^2 + 4s + 25}.$$

Coefficient matching gives $\omega_n = 5$, $2\zeta\omega_n = 4$, and therefore $\zeta = 0.4$. The poles are

$$p_{1,2} = -2 \pm j\sqrt{21}, \qquad \omega_d = \sqrt{21} \approx 4.583.$$

For a unit step,

$$y(t) = 1 - e^{-2t}\left[\cos(\sqrt{21}\,t) + \frac{2}{\sqrt{21}}\sin(\sqrt{21}\,t)\right].$$

The performance measures are

$$\begin{aligned} t_p &= \frac{\pi}{\sqrt{21}} \approx 0.686 \text{ s}, \\ M_p &= e^{-2\pi/\sqrt{21}} \approx 0.254, \qquad \text{PO} \approx 25.4\%, \\ t_s^{2\%} &\approx \frac{4}{2} = 2.00 \text{ s}, \end{aligned}$$

$$t_r^{0\text{–}100} = \frac{\pi - \cos^{-1}(0.4)}{\sqrt{21}} \approx 0.433 \text{ s}.$$

The envelope calculation gives $t_{\text{env}} \approx 2.00$ s in this example; the near agreement with $4/\sigma$ is not a general identity. The exact settling time should be found from the final band crossing when precise timing is required.

From Specifications to Regions in the Pole Plane Suppose a standard second-order target should satisfy PO $\leq 10\%$ and approximately $t_s^{2\%} \leq 2$ s. The overshoot requirement gives

$$\zeta \geq \frac{-\ln(0.10)}{\sqrt{\pi^2 + \ln^2(0.10)}} \approx 0.591.$$

The settling-time estimate gives $\sigma \gtrsim 2$. These suggest poles sufficiently far to the left and not too close in angle to the imaginary axis. These are performance targets, not yet a controller-design

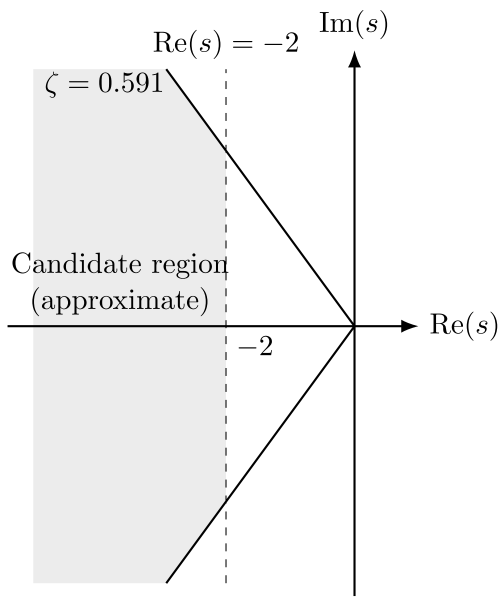

Figure 5: A schematic target region: to the left of the settling boundary and inside the damping-ratio wedge. Full-system verification remains necessary.

method. Later, root locus will help us determine whether a gain or compensator can place closed-loop poles in such a region. Zeros and additional poles can change the actual performance even when a chosen pair lies in the region.

6 Dominant Poles and Higher-Order Systems

Why Low-Order Models Can Be Useful A physical system may have many poles, but not all modes contribute equally over the time interval of interest. For distinct stable poles, a step response often has the form

$$y(t) = y_\infty + \sum_i A_i e^{p_i t}.$$

Modes with more negative real parts decay faster; those closer to the imaginary axis are candidates to dominate the late transient. For complex poles, decay rates are determined by their real parts, and a conjugate pair is treated as one real oscillatory mode.

A dominant-pole approximation requires both a separation of decay rates and a meaningful contribution from the retained modes. Pole locations alone do not determine the approximation error.

A Common Approximation Procedure Use the following sequence:

1. Check stability before discarding modes. Never discard an unstable pole merely because it is far from the origin.
2. Identify the slow real pole or conjugate pair and compare pole real parts. A separation of roughly a factor of five is a useful starting heuristic, not a theorem.
3. Check residues and nearby zeros for the input and initial conditions of interest. A nearly canceled slow pole may contribute little, while a fast mode with a large residue can affect the early response.
4. Replace fast factors by their low-frequency values, preserve the DC gain, and compare the approximate and full responses over the relevant time interval.

Example: A Fast Additional Pole Consider

$$G(s) = \frac{100}{(s+10)(s^2+2s+10)} = \frac{10}{s+10}\,\frac{10}{s^2+2s+10}.$$

Its poles are -10 and $-1 \pm j3$, and $G(0) = 1$. Since the extra pole decays as e^{-10t} while the pair decays as e^{-t}, a possible approximation is

$$\boxed{G_{\text{approx}}(s) = \frac{10}{s^2+2s+10}.}$$

This follows by replacing $10/(s+10)$ by 1, not by simply deleting $(s+10)$ and leaving the numerator unchanged. The exact unit-step response is

$$y(t) = 1 - \frac{1}{9}e^{-10t} - \frac{8}{9}e^{-t}\cos 3t - \frac{2}{3}e^{-t}\sin 3t.$$

The approximate response is

$$y_{\text{approx}}(t) = 1 - e^{-t}\left(\cos 3t + \frac{1}{3}\sin 3t\right).$$

Both have final value 1, and both contain the same slow oscillatory mode. However, the oscillatory coefficients differ: the extra pole changes the complete transient, not just an isolated early exponential term.

7 Effect of Zeros

Pole locations determine the available natural modes, but the numerator determines how strongly those modes appear in a particular response. We therefore examine what changes when the poles are fixed and a zero is added.

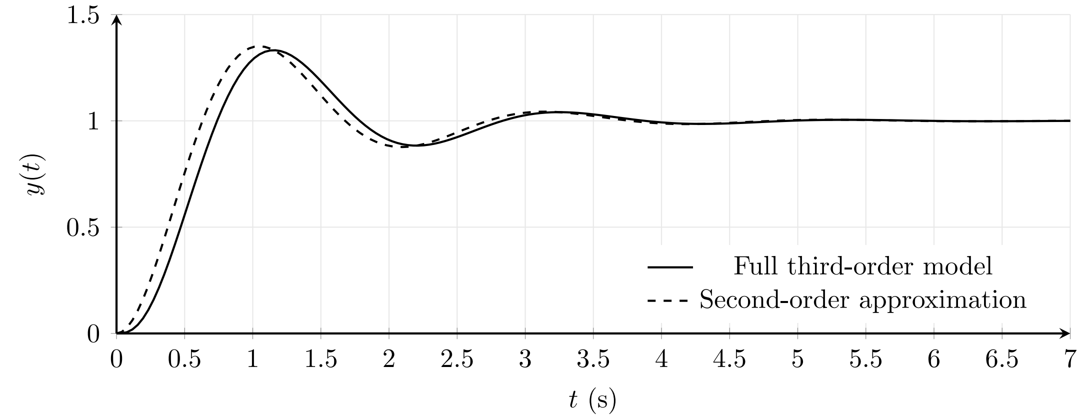

Figure 6: Retaining a dominant pair preserves much of the response shape, but the approximation is not exact.

A Zero Changes Modal Coefficients Compare two models with equal DC gain and equal poles:

$$G_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad G_2(s) = \left(1 + \frac{s}{z}\right) G_1(s), \qquad z > 0.$$

The second model has a zero at $s = -z$. For a unit step,

$$Y_2(s) = \left(1 + \frac{s}{z}\right) Y_1(s).$$

Because $y_1(0) = 0$, $\mathcal{L}\{\dot{y}_1(t)\} = sY_1(s)$, and therefore

$$\boxed{y_2(t) = y_1(t) + \frac{1}{z}\dot{y}_1(t).}$$

When y_1 is rising, its derivative is positive, so the added term raises the response. When y_1 is falling, the added term is negative. The final values agree because $\dot{y}_1(t) \to 0$.

Example: A Left-Half-Plane Zero Take

$$G_1(s) = \frac{4}{s^2 + 2s + 4}, \qquad G_2(s) = \frac{4(s+1)}{s^2 + 2s + 4}.$$

Both have poles $-1 \pm j\sqrt{3}$ and DC gain 1. Yet

$$y_2(t) = y_1(t) + \dot{y}_1(t), \qquad \dot{y}_1(t) = \frac{4}{\sqrt{3}} e^{-t} \sin(\sqrt{3}\,t),$$

so their peak values and peak times differ. The standard no-zero overshoot formula applies to G_1, not directly to G_2. A far-left zero has large z, making the derivative contribution smaller for this family. A nearby zero can make the standard second-order formulas inaccurate. This lecture focuses on this basic effect; a detailed treatment of other zero configurations comes later.

Question for Discussion

A third-order system has a pole pair whose ζ predicts 10% overshoot. Is 10% guaranteed for the full system? What information is missing?

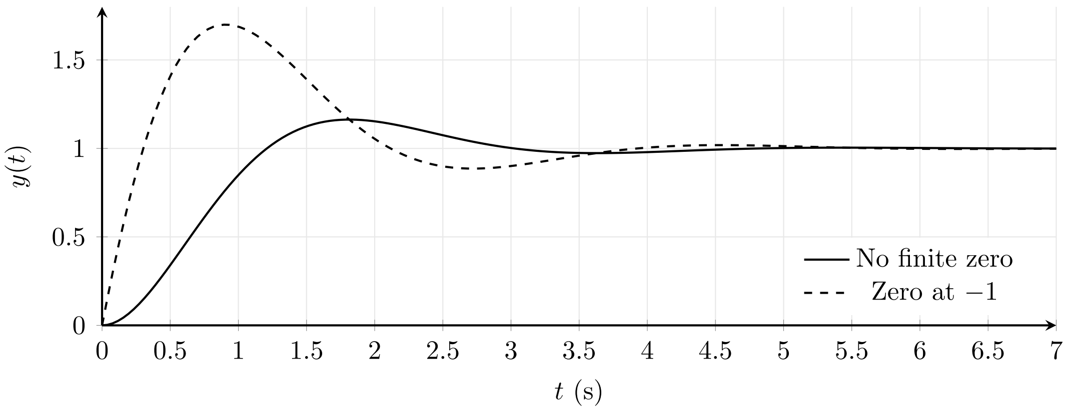

Figure 7: Identical poles do not imply identical step-response performance. The numerator matters.

8 Main Takeaways

1. The complete response is the sum of zero-input and zero-state responses; this decomposition is different from the transient–steady-state decomposition.
2. Transfer-function poles determine the available natural modes. Their real parts set decay rates, and their imaginary parts set oscillation frequencies.
3. First- and second-order models provide standard formulas for rise time, peak time, overshoot, and settling time.
4. Performance formulas require their modeling assumptions. Extra poles and zeros can change modal coefficients and invalidate a direct low-order interpretation.
5. A dominant-pole approximation requires both a separation of decay rates and a meaningful contribution from the retained modes; it should be checked against the full response.

Homework

Unless stated otherwise, assume zero initial conditions and a unit-step input or reference, as appropriate. Show the reasoning leading to each result. Use numerical software only as a check after the hand calculation.

Problem 1. (Modes and complete response.) A system satisfies

$$\dot{y}(t) + 3y(t) = 6u(t), \qquad y(0) = 1, \qquad u(t) = 1 \ (t \geq 0).$$

Find the complete response. Identify its zero-input, zero-state, transient, and steady-state components. Explain why the two decompositions are different.

Problem 2. (First-order measurements.) Let $G(s) = 8/(4s + 2)$. Find the DC gain, time constant, and pole. Derive the response to a step of amplitude 2. Calculate the exact 2% settling time and the 10–90% rise time.

Problem 3. (Second-order performance.) Let

$$T(s) = \frac{36}{s^2 + 6s + 36}.$$

Find $\omega_n, \zeta, \sigma, \omega_d$, and the poles. Derive the unit-step response. Calculate percent overshoot, peak time, first 0–100% rise time, the approximate 2% settling time, and the envelope-based sufficient settling time.

Problem 4. (Reverse calculation from a specification.) For a no-zero standard second-order model, require at most 5% overshoot and an estimated 2% settling time at most 2 s. Find the corresponding minimum damping ratio and minimum decay rate. Choose a pole pair satisfying both targets and write a unity-DC-gain transfer function. Explain why these calculations alone would not certify a general higher-order system.

Problem 5. (Dominant poles.) Consider

$$G(s) = \frac{200}{(s + 20)(s^2 + 2s + 10)}.$$

Find its poles and DC gain. Propose a second-order approximation that preserves DC gain. Identify two reasons why the approximation should still be checked against the full response.

Problem 6. (A zero with fixed poles.) Let

$$G_1(s) = \frac{9}{s^2 + 3s + 9}, \qquad G_2(s) = \left(1 + \frac{s}{2}\right) G_1(s).$$

Show that their unit-step responses satisfy $y_2(t) = y_1(t) + \dot{y}_1(t)/2$. Do they have the same poles and final value? Do they necessarily have the same overshoot? Compare their initial slopes.