ME 41100: System Dynamics and Control
Lecture 4: Block Diagrams and Feedback Interconnections

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Learning Objectives. By the end of this lecture, you should be able to:

- interpret blocks, signal arrows, summing points, and pickoff points in a block diagram;
- combine transfer function models in series, parallel, and feedback interconnections;
- derive the closed-loop transfer function of a standard feedback system from signal equations;
- distinguish the controller, plant, sensor, forward path, feedback path, and loop transfer function;
- determine transfer functions between different signal pairs in the same interconnected system;
- reduce a simple multiloop block diagram and verify the result using signal equations.

1 Where We Are in the Course

Lecture 3 introduced the transfer function as an algebraic input–output representation of an LTI system. For one block,

$$Y(s) = G(s)U(s), \qquad G(s) = \frac{Y(s)}{U(s)},$$

under zero initial conditions. The next question is how to connect several such component models to represent a complete control system.

How do we represent interconnected dynamic systems, and how do we derive the transfer function between signals of interest in a feedback loop?

The main workflow of this lecture is

$$\boxed{\text{Component Models}} \longrightarrow \boxed{\text{Block Diagram}} \longrightarrow \boxed{\text{Signal Equations}} \longrightarrow \boxed{\text{Closed-Loop Model}}.$$

This lecture continues the *representation* step in the broader course workflow:

$$\text{Physical System} \rightarrow \text{Dynamic Model} \rightarrow \boxed{\text{Represent}} \rightarrow \text{Analyze} \rightarrow \text{Design} \rightarrow \text{Verify}.$$

A block diagram is a graphical representation of mathematical relations among signals. It is not a physical drawing of the hardware.

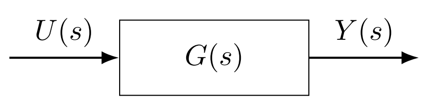

Figure 1: A transfer function block represents the relation $Y(s) = G(s)U(s)$.

2 Basic Elements of a Block Diagram

Four elements appear repeatedly.

Blocks and Signal Arrows. A block maps an input signal to an output signal. If the block is labeled $G(s)$, then

$$Y(s) = G(s)U(s).$$

The arrow indicates the direction in which the signal relation is read.

Summing Points. A summing point forms an algebraic sum of incoming signals. For example,

$$E(s) = R(s) - Y_m(s).$$

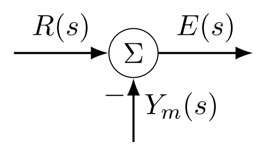

Figure 2: A summing point forming the error $E(s) = R(s) - Y_m(s)$.

Signals entering the same summing point must represent compatible physical quantities.

Pickoff Points. A pickoff, or takeoff, point sends the same signal to more than one destination. It does not create a new signal or change its value.

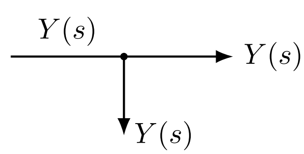

Figure 3: A pickoff point copies the same signal into multiple paths.

3 Elementary Interconnections

Series or Cascade Connection. Consider two systems connected in series:

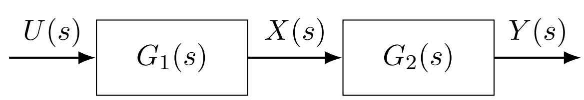

Figure 4: Series interconnection.

The signal equations are

$$X(s) = G_1(s)U(s), \qquad Y(s) = G_2(s)X(s).$$

Therefore,

$$\boxed{\frac{Y(s)}{U(s)} = G_2(s)G_1(s).}$$

This multiplication rule applies to the input–output models represented by the blocks. When physical components are directly connected, loading may change the component dynamics; the component models must remain valid under the interconnection.

Parallel Connection. Consider two systems connected in parallel, with their outputs added:

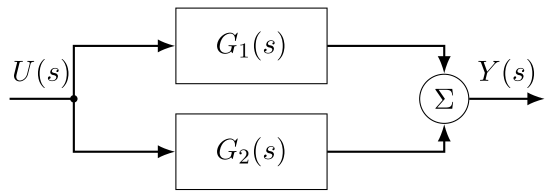

Figure 5: Parallel interconnection.

The signal equation is

$$Y(s) = G_1(s)U(s) + G_2(s)U(s).$$

Therefore,

$$\boxed{\frac{Y(s)}{U(s)} = G_1(s) + G_2(s).}$$

If one branch enters the summing point with a negative sign, the corresponding transfer functions subtract.

4 The Standard Feedback Interconnection

A feedback control system usually contains a controller $C(s)$, a plant $P(s)$, and a sensor or feedback element $H(s)$. The measured output is compared with the reference to form the error. The transfer function from the error $E(s)$ to the plant output $Y(s)$ is the **forward-path transfer function**

$$G(s) = C(s)P(s),$$

while $H(s)$ is the **feedback-path transfer function**.

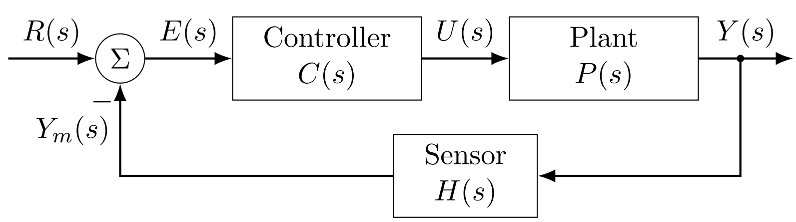

Figure 6: Standard negative-feedback interconnection.

The signal equations are

$$E(s) = R(s) - H(s)Y(s), \qquad U(s) = C(s)E(s), \qquad Y(s) = P(s)U(s).$$

Substituting from one equation into the next gives

$$Y(s) = C(s)P(s)\big[R(s) - H(s)Y(s)\big].$$

Therefore,

$$\big[1 + C(s)P(s)H(s)\big]Y(s) = C(s)P(s)R(s),$$

and the **closed-loop transfer function** from reference to output is

$$\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)}.}$$

Define the **loop transfer function**

$$\boxed{L(s) = G(s)H(s) = C(s)P(s)H(s).}$$

At the summing point, $R(s)$ and $H(s)Y(s)$ must have the same physical units because they are subtracted. Therefore, after a signal travels once around the loop and returns to the summing point, its units must be unchanged. Thus the product $L(s) = C(s)P(s)H(s)$ is dimensionless.

For negative feedback, the closed-loop transfer function has the denominator $1 + L(s)$. The closed-loop poles are the values of s that make this denominator zero, so the **closed-loop characteristic equation** is

$$\boxed{1 + L(s) = 0.}$$

For example, if

$$L(s) = \frac{N_L(s)}{D_L(s)},$$

then

$$1 + L(s) = \frac{D_L(s) + N_L(s)}{D_L(s)},$$

and the closed-loop characteristic polynomial is $D_L(s) + N_L(s)$. Its roots are the **closed-loop poles**. We will use this relationship later to study stability, root locus, and frequency-domain methods.

Unity Feedback. If the output is fed back directly, then $H(s) = 1$. This is called **unity feedback**, and

$$\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}.}$$

Positive Feedback. For positive feedback, the summing relation is $E(s) = R(s) + H(s)Y(s)$. The same signal-equation calculation gives

$$\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 - C(s)P(s)H(s)}.}$$

Thus the sign at the summing point determines the sign of the loop term in the denominator. Positive feedback does not by itself guarantee instability; stability is determined by the resulting closed-loop poles.

5 One Diagram, Many Transfer Functions

A feedback system is not described by only one transfer function. Different choices of input and output define different input–output maps.

For the negative-feedback system in Figure 6, the **closed-loop transfer function** is

$$\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)}.$$

Since $E(s) = R(s) - H(s)Y(s)$, the **error transfer function** is

$$\boxed{\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)P(s)H(s)}.}$$

Since $U(s) = C(s)E(s)$, the **reference-to-control transfer function** is

$$\boxed{\frac{U(s)}{R(s)} = \frac{C(s)}{1 + C(s)P(s)H(s)}.}$$

These transfer functions describe different signals in the same closed-loop system.

Disturbance Input and Superposition. Suppose a disturbance $D(s)$ enters at the plant input.

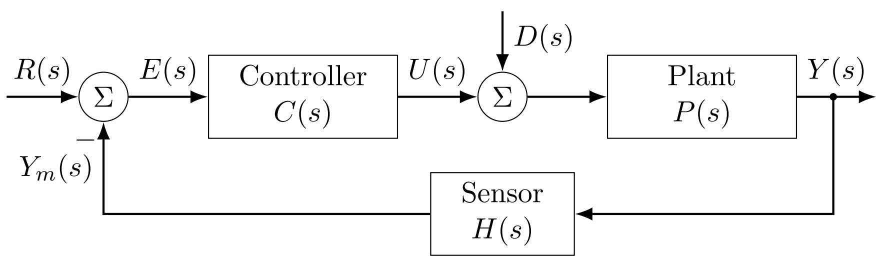

Figure 7: A feedback loop with an additive plant-input disturbance.

Then

$$Y(s) = P(s)\big[U(s) + D(s)\big], \qquad U(s) = C(s)\big[R(s) - H(s)Y(s)\big].$$

Substitution gives

$$\big[1 + C(s)P(s)H(s)\big]Y(s) = C(s)P(s)R(s) + P(s)D(s).$$

Hence

$$\boxed{Y(s) = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)}R(s) + \frac{P(s)}{1 + C(s)P(s)H(s)}D(s).}$$

By superposition, the transfer function from one independent input is found by setting the other independent inputs to zero. Thus, with $D(s) = 0$ when computing $Y(s)/R(s)$ and $R(s) = 0$ when computing $Y(s)/D(s)$,

$$\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)H(s)}, \qquad \frac{Y(s)}{D(s)} = \frac{P(s)}{1 + C(s)P(s)H(s)}.}$$

One interconnected system generally contains several transfer functions. Always specify which signal is treated as the input and which signal is treated as the output.

6 Block-Diagram Reduction

Simple diagrams can often be reduced by repeatedly replacing familiar substructures with equivalent blocks. A systematic procedure is:

1. specify the desired input and output;
2. identify the innermost series, parallel, or feedback substructure;
3. replace that substructure with its equivalent transfer function;
4. redraw the remaining diagram, preserving all summing and pickoff points;
5. repeat until a single equivalent block remains.

Signal equations are fundamental; block-diagram reduction is a shortcut. If a reduction step is unclear, write the internal signal equations and eliminate the internal variables.

Example: Two Nested Feedback Loops. Consider the interconnection below. The inner feedback path contains $H_2(s)$, while $H_1(s)$ closes an outer loop around the entire forward path.

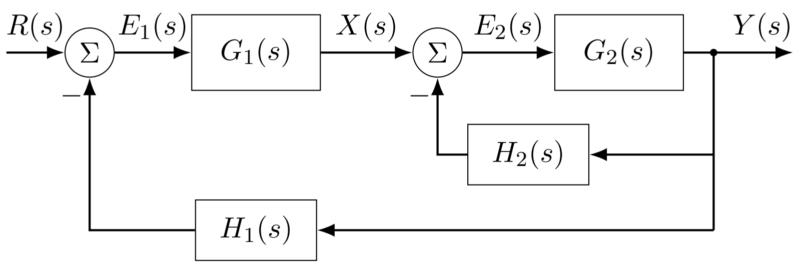

Figure 8: A system with inner and outer negative-feedback loops.

First reduce the inner loop:

$$G_{\text{inner}}(s) = \frac{G_2(s)}{1 + G_2(s)H_2(s)}.$$

The outer loop then has forward path $G_1(s)G_{\text{inner}}(s)$ and feedback path $H_1(s)$. Therefore,

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{G_1(s)G_{\text{inner}}(s)}{1 + G_1(s)G_{\text{inner}}(s)H_1(s)} \\ &= \boxed{\frac{G_1(s)G_2(s)}{1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)}}.\end{aligned}$$

To verify the result directly, let $E_1(s)$ and $E_2(s)$ denote the outputs of the two summing points, and let $X(s)$ denote the output of $G_1(s)$. The signal equations are

$$E_1(s) = R(s) - H_1(s)Y(s), \qquad X(s) = G_1(s)E_1(s),$$

$$E_2(s) = X(s) - H_2(s)Y(s), \qquad Y(s) = G_2(s)E_2(s).$$

Eliminating the internal signals gives

$$Y(s) = G_2(s)\Big[G_1(s)\big(R(s) - H_1(s)Y(s)\big) - H_2(s)Y(s)\Big]$$

$$= G_1(s)G_2(s)R(s) - G_1(s)G_2(s)H_1(s)Y(s) - G_2(s)H_2(s)Y(s),$$

and therefore

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)},$$

which agrees with the result obtained by successive reduction.

Moving a Summing Point: Preserve the Signal Relation. There is no need to memorize a long list of diagram-transformation rules. The rule is simply to preserve the same signal equation. For example,

$$Y(s) = G(s)\big[X(s) + D(s)\big] = G(s)X(s) + G(s)D(s).$$

If a summing point originally appears before $G(s)$, then moving it after $G(s)$ requires multiplying the side input by $G(s)$. The algebra determines the correct transformation.

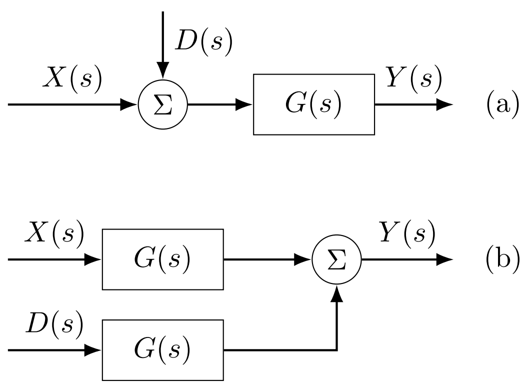

Figure 9: Moving a summing point from before a block to after it requires multiplying the side input by the same transfer function.

7 Mechanical Example: Proportional Position Feedback

Return to the mass–spring–damper system used in Lectures 1–3:

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t).$$

Its force-to-position transfer function is

$$P(s) = \frac{Q(s)}{U(s)} = \frac{1}{ms^2 + bs + k}.$$

Suppose the control law is proportional feedback,

$$u(t) = K\big(r(t) - q(t)\big), \qquad K > 0.$$

The corresponding unity-feedback block diagram is

Using the unity-feedback formula,

$$\frac{Q(s)}{R(s)} = \frac{KP(s)}{1 + KP(s)}$$

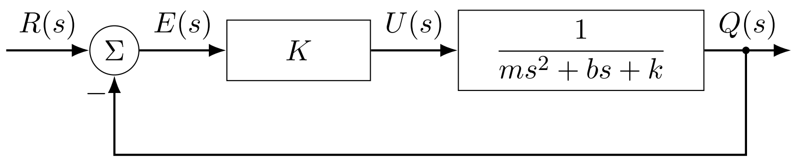

Figure 10: Proportional position feedback for the mass–spring–damper system.

$$= \boxed{\frac{K}{ms^2 + bs + k + K}}.$$

Substituting the control law directly into the mechanical model gives

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = K\big[r(t) - q(t)\big],$$

and therefore

$$\boxed{m\ddot{q}(t) + b\dot{q}(t) + (k + K)q(t) = Kr(t).}$$

Proportional position feedback acts like an additional stiffness K. It changes the effective stiffness from k to $k + K$, while the mass and damping coefficients remain unchanged.

Thus feedback creates a new closed-loop dynamical system. The open-loop denominator

$$ms^2 + bs + k$$

is replaced by the closed-loop denominator

$$ms^2 + bs + k + K.$$

In the next lecture, we will study how the poles of such denominators determine the dynamic response.

Question for Discussion

Without solving for $q(t)$, which terms in the governing differential equation are changed by proportional feedback, and what is their physical interpretation? Which properties of the time response require further analysis of the closed-loop poles?

8 A Short MATLAB Check

MATLAB can construct the same feedback interconnection directly after numerical parameter values have been assigned. For example, let

$$m = 1, \qquad b = 2, \qquad k = 5, \qquad K = 10.$$

Then:

```
m = 1; b = 2; k = 5; K = 10;
s = tf('s');
P = 1/(m*s^2 + b*s + k);
C = K;
Tcl = feedback(C*P,1)
```

The result is

$$T_{\mathrm{cl}}(s) = \frac{10}{s^2 + 2s + 15},$$

which agrees with the general expression

$$\frac{Q(s)}{R(s)} = \frac{K}{ms^2 + bs + k + K}.$$

The software performs the same interconnection algebra derived above; it does not replace the derivation.

9 Main Takeaways

- Block diagrams represent mathematical relations among signals, not the physical geometry of a system.
- Blocks transform signals, summing points add or subtract compatible signals, and pickoff points copy a signal without changing it.
- Series transfer functions multiply, parallel transfer functions add or subtract, and negative feedback produces the denominator $1 + L(s)$.
- The characteristic equation $1 + L(s) = 0$, where $L(s) = C(s)P(s)H(s)$, determines the closed-loop poles.
- One interconnected system can define several transfer functions; when multiple independent inputs are present, the others are set to zero one at a time.
- Block diagrams should be reduced from the innermost substructure outward. Signal equations provide a reliable derivation and verification method when a reduction step is unclear.
- Feedback changes the dynamics of the system; in the mechanical example, proportional position feedback increases the effective stiffness from k to $k + K$.

Homework

1. In your own words, explain the mathematical roles of a block, a signal arrow, a summing point, and a pickoff point. Why must signals entering the same summing point represent compatible physical quantities?

2. Let

$$G_1(s) = \frac{2}{s+1}, \qquad G_2(s) = \frac{5}{s+3}.$$

   Find the equivalent transfer function when the two blocks are connected (a) in series and (b) in parallel with their outputs added. Derive both results from internal signal equations.

3. For a standard negative-feedback system with controller $C(s)$, plant $P(s)$, and feedback element $H(s)$, begin with the signal equations and derive

$$\frac{Y(s)}{R(s)}, \qquad \frac{E(s)}{R(s)}, \qquad \frac{U(s)}{R(s)}.$$

   State the conventional name and physical role of each transfer function. Repeat the derivation of $Y(s)/R(s)$ for positive feedback.

4. Consider a negative-feedback system with

$$C(s) = \frac{s+2}{s+5}, \qquad P(s) = \frac{3}{s(s+4)}, \qquad H(s) = 1.$$

Determine the closed-loop transfer function $Y(s)/R(s)$ and write its denominator as a polynomial in s. Identify the corresponding closed-loop characteristic equation.

5. A disturbance $D(s)$ is added at the plant input of a negative-feedback system. Starting from

$$Y(s) = P(s)\big(U(s) + D(s)\big), \qquad U(s) = C(s)\big(R(s) - H(s)Y(s)\big),$$

express $Y(s)$ as a sum of its response to $R(s)$ and its response to $D(s)$. Clearly state which independent input must be set to zero when defining each transfer function.

6. For the nested-feedback system in Figure 8, let

$$G_1(s) = \frac{2}{s+1}, \qquad G_2(s) = \frac{3}{s+2}, \qquad H_1(s) = H_2(s) = 1.$$

Derive $Y(s)/R(s)$ in two ways: (a) by reducing the inner loop first and (b) by writing and eliminating the internal signals. Verify that the two expressions agree, and express the final result as a single rational function of s.

7. Consider the signal relation

$$Y(s) = G(s)X(s) + D(s),$$

in which the summing point is located after $G(s)$. Move the summing point to a position before $G(s)$, draw the equivalent block diagram, and determine what transfer function must be inserted in the disturbance path. Verify the result using the signal equations.

8. Consider the mass–spring–damper system

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t)$$

with proportional position-feedback control law

$$u(t) = K\big[r(t) - q(t)\big], \qquad K > 0.$$

(a) Derive the closed-loop transfer function $Q(s)/R(s)$.

(b) Substitute the control law into the mechanical model and derive the closed-loop differential equation.

(c) Explain how proportional position feedback changes the effective mass, damping, and stiffness.

(d) For $m = 2$, $b = 3$, $k = 4$, and $K = 8$, verify the transfer function using MATLAB's `feedback` command.