ME 41100: System Dynamics and Control
Lecture 3: The Laplace Transform and Transfer Functions

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Learning Objectives. By the end of this lecture, you should be able to:

- explain why the Laplace transform is useful for representing and analyzing LTI dynamic systems;
- compute Laplace transforms of common causal signals using the definition, standard transform pairs, and basic properties;
- transform first- and second-order differential equations while correctly accounting for initial conditions;
- recover simple time-domain responses using inverse Laplace transforms and partial-fraction expansion;
- define and derive a transfer function for a specified input–output pair, and explain why its definition uses zero initial conditions;
- identify system order, poles, zeros, and DC gain from a transfer function, and explain how the choice of output affects the transfer function.

1 Where We Are in the Course

Lecture 2 ended with a dynamic model written as a *differential equation*. Setting the disturbance $d(t) = 0$ for simplicity, the mass–spring–damper model becomes

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t). \tag{1}$$

The model is physically meaningful, but we now want a representation that is easier to manipulate when systems are interconnected and feedback loops are formed.

How can we transform a differential-equation model into a representation that is easier to analyze and interconnect? How can we convert a differential-equation model into an algebraic input–output representation?

The main workflow of this lecture is

$$\boxed{\text{Differential Equation}} \longrightarrow \boxed{\text{Laplace Transform}} \longrightarrow \boxed{\text{Transfer Function}}.$$

This is the *representation* step in the broader course workflow:

$$\text{Physical System} \to \text{Dynamic Model} \to \boxed{\text{Represent}} \to \text{Analyze} \to \text{Design} \to \text{Verify}.$$

This lecture develops the representation step by converting an ODE model into a transfer function.

2 Why Use the Laplace Transform?

The central computational advantage is simple:

The Laplace transform converts differentiation in time into multiplication by the complex variable s.

For example, if initial conditions are temporarily ignored, Equation (1) becomes

$$ms^2Q(s) + bsQ(s) + kQ(s) = U(s),$$

which is an **algebraic equation** in the s-domain rather than a differential equation in time.

This change of representation is useful because algebraic equations are easier to combine, rearrange, and manipulate. Later, the same representation will make feedback interconnections, poles and zeros, stability analysis, frequency response, and controller design much more systematic.

Why Exponentials Are Special. There is a deeper reason that Laplace methods work so naturally for linear time-invariant systems. Consider the exponential signal

$$e^{st},$$

where s may be complex. Differentiation preserves its form:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{st} = se^{st}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{st} = s^2e^{st}.$$

Thus differentiation changes only the multiplier, not the basic shape of the signal.

For the homogeneous second-order equation

$$a_2\ddot{y} + a_1\dot{y} + a_0y = 0, \tag{2}$$

try a solution $y(t) = Ae^{st}$. Substitution gives

$$\left(a_2s^2 + a_1s + a_0\right)Ae^{st} = 0.$$

A nonzero solution therefore requires

$$a_2s^2 + a_1s + a_0 = 0. \tag{3}$$

This is the **characteristic equation**. We will return to its roots when we study poles and dynamic response.

This connection among exponential signals, linear differential equations, and characteristic roots is one reason Laplace-domain methods are especially effective for LTI systems.

Question for Discussion

Why does replacing differentiation by multiplication make interconnections and feedback calculations easier? What operation would be harder to perform directly on differential equations?

3 Definition of the Laplace Transform

For an engineering signal $f(t)$ considered for $t \geq 0$, we define its one-sided Laplace transform by

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t.} \tag{4}$$

The variable s is complex:

$$s = \sigma + j\omega, \qquad j^2 = -1, \tag{5}$$

where σ and ω have units of inverse time.

Brief complex-number review. For $s = \sigma + j\omega$,

$$s = |s|e^{j\angle s}, \qquad |s| = \sqrt{\sigma^2 + \omega^2}, \qquad \angle s = \operatorname{atan2}(\omega, \sigma).$$

For $s \neq 0$, it follows that

$$\left|\frac{1}{s}\right| = \frac{1}{|s|}, \qquad \angle\left(\frac{1}{s}\right) = -\angle s.$$

Remark. The lower limit 0^- means that the integral begins just before $t = 0$. This convention includes any impulse located at $t = 0$ and uses the pre-switch value $f(0^-)$ in the Laplace-transform formulas for derivatives. For ordinary signals that are continuous at $t = 0$ and contain no impulse there, we simply write $f(0)$.

The Laplace transform maps a time-domain function $f(t)$ to an s-domain function $F(s)$:

$$f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad F(s).$$

In this course, we will assume the signals of interest have Laplace transforms. A detailed study of regions of convergence is not needed for our present control objectives.

4 Two Basic Laplace Transforms from the Definition

We denote the unit step by $\mathbf{1}(t)$:

$$\mathbf{1}(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

It is useful to derive a few transforms directly before relying on a table.

Example 1: Unit Step. Using Equation (4),

$$\mathcal{L}\{\mathbf{1}(t)\} = \int_{0^-}^{\infty} e^{-st}\,\mathrm{d}t = \left[-\frac{1}{s}e^{-st}\right]_{0^-}^{\infty} = \frac{1}{s},$$

provided the real part of s is positive. Therefore,

$$\boxed{\mathbf{1}(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s}.} \tag{6}$$

Example 2: Decaying Exponential. For $a > 0$, let

$$f(t) = e^{-at}\mathbf{1}(t).$$

Then

$$F(s) = \int_{0^-}^{\infty} e^{-at}e^{-st}\,\mathrm{d}t = \int_{0^-}^{\infty} e^{-(s+a)t}\,\mathrm{d}t = \frac{1}{s+a}.$$

Hence

$$\boxed{e^{-at}\mathbf{1}(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+a}.} \tag{7}$$

The parameter a is related to the time constant by

$$T = \frac{1}{a},$$

so the same transform pair can be written as

$$e^{-t/T}\mathbf{1}(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+1/T} = \frac{T}{Ts+1}.$$

The second form will appear naturally when we analyze first-order systems.

5 Common Laplace Transform Pairs

The following pairs are used repeatedly in control analysis.

Table 1: Common one-sided Laplace transform pairs.

Time-domain signal	**s-domain transform**	**Comment**
$\delta(t)$	1	unit impulse
$\mathbf{1}(t)$	$\frac{1}{s}$	unit step
$t\mathbf{1}(t)$	$\frac{1}{s^2}$	unit ramp
$t^n\mathbf{1}(t)$	$\frac{n!}{s^{n+1}}$	polynomial, $n = 0, 1, 2, \ldots$
$e^{-at}\mathbf{1}(t)$	$\frac{1}{s+a}$	decaying exponential, $a > 0$
$\sin(\omega t)\mathbf{1}(t)$	$\frac{\omega}{s^2+\omega^2}$	sinusoid
$\cos(\omega t)\mathbf{1}(t)$	$\frac{s}{s^2+\omega^2}$	sinusoid

The unit impulse $\delta(t)$ is an idealized signal concentrated at $t = 0$ and satisfying

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1, \qquad \int_{-\infty}^{\infty} \delta(t)f(t)\,\mathrm{d}t = f(0).$$

The second identity holds for any f that is continuous at $t = 0$. Because the lower limit of the one-sided transform is 0^-, an impulse located at $t = 0$ is included, and therefore

$$\mathcal{L}\{\delta(t)\} = 1.$$

We will use impulses later when interpreting impulse response and transfer functions; for now, the main Laplace pairs to remember are the step, exponential, and basic sinusoidal forms.

Question for Discussion

Without integrating, what do you expect the Laplace transform of $5e^{-2t}\mathbf{1}(t) + 3\mathbf{1}(t)$ to be? Which property justifies your answer?

6 Properties of the Laplace Transform

Linearity. If

$$\mathcal{L}\{f_1(t)\} = F_1(s), \qquad \mathcal{L}\{f_2(t)\} = F_2(s),$$

then

$$\boxed{\mathcal{L}\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s).} \tag{8}$$

This property is essential because the differential equations studied in the first part of ME 41100 are linear.

Differentiation. The most important Laplace property for solving ODEs is

$$\boxed{\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0^-).} \tag{9}$$

It follows directly from integration by parts. Starting from the definition,

$$\begin{aligned}
\mathcal{L}\{\dot{f}(t)\} &= \int_{0^-}^{\infty} \dot{f}(t)e^{-st}\,\mathrm{d}t \\
&= \left[f(t)e^{-st}\right]_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t \\
&= -f(0^-) + sF(s),
\end{aligned}$$

where we assume $f(t)e^{-st} \to 0$ as $t \to \infty$ in the region where the transform exists. This derivation explains exactly why the initial condition appears.

Apply Equation (9) once more to obtain the second-derivative property:

$$\boxed{\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - sf(0^-) - \dot{f}(0^-).} \tag{10}$$

More generally,

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dot{f}(0^-) - \cdots - f^{(n-1)}(0^-). \tag{11}$$

Notice what has happened: the derivatives have become powers of s, while the initial conditions appear explicitly as algebraic terms.

The Laplace transform does not discard initial conditions. It moves them from a differential equation into algebraic terms in the s-domain.

Integration. If

$$g(t) = \int_{0^-}^{t} f(\xi)\,\mathrm{d}\xi,$$

then $\dot{g}(t) = f(t)$ and $g(0^-) = 0$. The differentiation property therefore gives

$$sG(s) = F(s),$$

and hence

$$\boxed{\mathcal{L}\{g(t)\} = \frac{F(s)}{s}.} \tag{12}$$

7 Inverse Laplace Transforms and Partial-Fraction Expansion

To recover a time-domain signal from its Laplace transform, we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}, \qquad t \geq 0.$$

When $F(s)$ is a proper rational function—that is, when the numerator degree is smaller than the denominator degree—**partial-fraction expansion** rewrites it as a sum of simpler terms that can be inverted using Table 1.

For example, consider the proper rational function

$$F(s) = \frac{2}{(s+3)(s+5)}.$$

Because its denominator has two distinct linear factors, seek an expansion of the form

$$\frac{2}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}.$$

Multiplying by $(s+3)(s+5)$ gives

$$2 = A(s+5) + B(s+3).$$

Setting $s = -3$ gives $A = 1$; setting $s = -5$ gives $B = -1$. Thus

$$F(s) = \frac{1}{s+3} - \frac{1}{s+5},$$

and therefore

$$\boxed{f(t) = e^{-3t} - e^{-5t}, \qquad t \geq 0.} \tag{13}$$

The basic procedure is to factor the denominator, expand the rational function into simpler terms, determine the unknown coefficients, and then use known transform pairs.

We now apply this procedure to a first-order system with a nonzero initial condition and a step input.

8 Solving a First-Order ODE in the s-Domain

Consider the first-order model introduced in Lecture 2:

$$T\dot{y}(t) + y(t) = Ku(t), \qquad T > 0. \tag{14}$$

Suppose the initial condition is

$$y(0^-) = y_0.$$

Take the Laplace transform of both sides:

$$T\big[sY(s) - y_0\big] + Y(s) = KU(s), \tag{15}$$

$$(Ts+1)Y(s) = KU(s) + Ty_0. \tag{16}$$

Therefore,

$$\boxed{Y(s) = \frac{K}{Ts+1}U(s) + \frac{Ty_0}{Ts+1}.} \tag{17}$$

Equation (17) separates two components of the response:

- the first term is the input-driven, or zero-state, response;
- the second term is the initial-condition, or zero-input, response.

This distinction will become important when we define the transfer function.

Step-Input Response. Let the input be a constant step of magnitude U_0:

$$u(t) = U_0\mathbf{1}(t), \qquad U(s) = \frac{U_0}{s}.$$

Then

$$Y(s) = \frac{KU_0}{s(Ts+1)} + \frac{Ty_0}{Ts+1}. \tag{18}$$

To invert the first term, write

$$\frac{1}{s(Ts+1)} = \frac{A}{s} + \frac{B}{Ts+1}.$$

Multiplying by $s(Ts+1)$ gives

$$1 = A(Ts+1) + Bs.$$

Setting $s = 0$ gives $A = 1$, and matching the coefficient of s gives $B = -T$. Hence

$$\frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+1/T}.$$

Also,

$$\frac{Ty_0}{Ts+1} = \frac{y_0}{s+1/T}.$$

Therefore, for $t \geq 0$,

$$y(t) = KU_0\left(1 - e^{-t/T}\right) + y_0e^{-t/T} = \boxed{KU_0 + \big(y_0 - KU_0\big)e^{-t/T}}. \tag{19}$$

This result shows clearly that the response consists of a steady component plus a decaying exponential transient.

Question for Discussion

From Equation (19), what happens as $t \to \infty$? Verify that $y(0^+) = y_0$. Why does the finite step input not cause an instantaneous jump in $y(t)$? How does increasing T change the speed of the response?

9 Transfer Functions

We now use the Laplace transform for a different purpose. Instead of solving for one particular response, we want a compact description of the *input–output relationship* of an LTI system.

Return to the first-order system

$$T\dot{y} + y = Ku.$$

Equation (17) showed that

$$Y(s) = \frac{K}{Ts+1}U(s) + \frac{Ty_0}{Ts+1}.$$

Setting $y_0 = 0$ isolates the zero-state response:

$$Y(s) = \frac{K}{Ts+1}U(s).$$

Under zero initial conditions, the ratio of the output and input transforms is therefore

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{K}{Ts+1}.} \tag{20}$$

Definition. For an **LTI system**, the **transfer function** from input $u(t)$ to output $y(t)$ is

$$G(s) := \left[\frac{Y(s)}{U(s)}\right]_{\text{under zero initial conditions}}.$$

Four points are essential:

1. A transfer function is an **input–output description**.
2. It is a ratio of Laplace transforms, not the pointwise ratio $y(t)/u(t)$.
3. Its definition uses **zero initial conditions**.
4. It depends on the chosen input and the chosen output.

The transfer function does *not* mean that real systems always start from rest. Zero initial conditions are used to define the system's input–output map separately from the response caused by a nonzero initial state. The following simple block will become the basic building block of feedback diagrams in Lecture 4.

$$U(s) \longrightarrow \boxed{\quad G(s) \quad} \longrightarrow Y(s)$$

Figure 1: Block-diagram representation of a transfer function: $Y(s) = G(s)U(s)$.

10 From a General Linear ODE to a Transfer Function

Consider the linear constant-coefficient input–output equation

$$\begin{aligned} a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) \\ = b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t). \end{aligned} \tag{21}$$

where the coefficients are constant, $a_n \neq 0$, and $b_m \neq 0$.

With zero system initial conditions and an input that is zero for $t < 0$, the one-sided Laplace transform gives

$$\begin{aligned} \big(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\big) Y(s) \\ = \big(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0\big) U(s). \end{aligned}$$

Therefore,

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}.} \tag{22}$$

Thus a linear constant-coefficient input–output ODE yields a rational transfer function of s. Most physical systems considered in this course have $m \leq n$, so their transfer functions are proper. A rational transfer function is **proper** when the degree of its numerator does not exceed the degree of its denominator.

11 Examples: Mass–Spring–Damper Systems

Return to the model from Lecture 2, with $d(t) = 0$:

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t). \tag{23}$$

Including Initial Conditions First. Let

$$q(0^-) = q_0, \qquad \dot{q}(0^-) = v_0.$$

Taking Laplace transforms gives

$$m\big[s^2 Q(s) - sq_0 - v_0\big] + b\big[sQ(s) - q_0\big] + kQ(s) = U(s).$$

Collecting terms,

$$\big(ms^2 + bs + k\big) Q(s) = U(s) + m(sq_0 + v_0) + bq_0. \tag{24}$$

Hence

$$Q(s) = \frac{1}{ms^2 + bs + k} U(s) + \frac{m(sq_0 + v_0) + bq_0}{ms^2 + bs + k}. \tag{25}$$

Again, the first term is the input-driven response and the second term is the initial-condition response.

Position as the Output. To define the transfer function, set the initial conditions to zero. Then

$$\big(ms^2 + bs + k\big)Q(s) = U(s),$$

and the transfer function from applied force to position is

$$\boxed{G_q(s) = \frac{Q(s)}{U(s)} = \frac{1}{ms^2 + bs + k}.} \tag{26}$$

Velocity as the Output. Suppose the physical plant is unchanged, but the measured output is velocity

$$v(t) = \dot{q}(t).$$

Under zero initial conditions,

$$V(s) = sQ(s).$$

Therefore,

$$\boxed{G_v(s) = \frac{V(s)}{U(s)} = \frac{s}{ms^2 + bs + k}.} \tag{27}$$

Changing from position measurement to velocity measurement does not change the plant ODE, but it changes the input–output transfer function.

Same physical dynamics, different input–output representation.

Question for Discussion

For the same mass–spring–damper system, what is the transfer function from applied force u to acceleration $a = \ddot{q}$ under zero initial conditions?

Rotational Analogue. Consider the rotational analogue

$$J\ddot{\theta}(t) + b_\theta\dot{\theta}(t) + k_\theta\theta(t) = \tau(t). \tag{28}$$

Let

$$\mathcal{T}(s) = \mathcal{L}\{\tau(t)\}.$$

Under zero initial conditions,

$$\big(Js^2 + b_\theta s + k_\theta\big)\Theta(s) = \mathcal{T}(s),$$

so

$$\boxed{\frac{\Theta(s)}{\mathcal{T}(s)} = \frac{1}{Js^2 + b_\theta s + k_\theta}.} \tag{29}$$

The mathematical structure is identical to the translational system after the substitutions

$$m \leftrightarrow J, \qquad b \leftrightarrow b_\theta, \qquad k \leftrightarrow k_\theta, \qquad q \leftrightarrow \theta, \qquad u \leftrightarrow \tau.$$

This is another example of the principle from Lecture 2: different physical systems can share the same mathematical dynamic structure.

12 Poles, Zeros, Order, and DC Gain

A transfer function is commonly written as

$$G(s) = \frac{N(s)}{D(s)}, \tag{30}$$

where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial.

Poles and Zeros. After any common factors have been removed:

- the **zeros** are the roots of $N(s) = 0$;
- the **poles** are the roots of $D(s) = 0$.

For example,

$$G(s) = \frac{s+2}{(s+1)(s+4)}$$

has

$$\text{zero: } s = -2, \qquad \text{poles: } s = -1, -4.$$

At this stage, we only identify poles and zeros. Later we will see that poles govern the natural response and stability, while poles and zeros together shape the input–output response and frequency-domain behavior.

System Order. For a proper transfer function after common factors have been removed, the degree of the denominator is the order of the minimal input–output model represented by $G(s)$. Thus

$$G_q(s) = \frac{1}{ms^2 + bs + k}$$

is second order, consistent with the second-order mechanical ODE from which it was derived.

DC Gain. The **DC gain** is the transfer function evaluated at $s = 0$, provided this value is finite:

$$\boxed{K_{\mathrm{DC}} \coloneqq G(0).} \tag{31}$$

For a stable LTI system with finite $G(0)$, a constant input of magnitude U_0 produces the steady output

$$y_{\mathrm{ss}} = G(0)U_0.$$

For the first-order system,

$$G(s) = \frac{K}{Ts+1} \quad \Longrightarrow \quad G(0) = K.$$

For the mass–spring–damper position transfer function,

$$G_q(0) = \frac{1}{k}.$$

This has a direct physical interpretation. Under a constant static force F, the spring relation gives

$$kq = F \quad \Longrightarrow \quad \frac{q}{F} = \frac{1}{k}.$$

Thus the s-domain quantity $G_q(0)$ agrees with the static mechanical compliance.

A transfer function is not merely symbolic algebra. Its order, poles, zeros, and DC gain retain useful information about the underlying input–output dynamics and their physical interpretation.

13 Using Transfer Functions: Procedure and Interpretation

From an ODE to a Transfer Function. When deriving a transfer function from a differential-equation model, use the following steps:

1. **Choose the input and output.** A transfer function is defined for a particular input–output pair.
2. **Write the linear constant-coefficient ODE** relating that input and output.
3. **Take the Laplace transform** of every term.
4. **Keep the initial-condition terms initially** so you can see where they enter.
5. **Impose the transfer-function conditions.** Set the system initial conditions to zero and, when input derivatives are present, assume that the input is zero for $t < 0$.
6. **Collect the output terms** on one side and the input terms on the other.
7. **Form the ratio** $G(s) = Y(s)/U(s)$.
8. **Check the result**: order, units, signs, limiting cases, and DC gain should make physical sense.

Question for Discussion

Why is "set all initial conditions to zero" a dangerous first step if the actual goal is to solve an initial-value problem rather than define a transfer function?

What a Transfer Function Does and Does Not Tell Us. A transfer function is powerful, but its meaning should be kept precise.

Table 2: What is encoded by a transfer function?

Question	Answer
What does it describe?	The zero-state input–output behavior of an LTI system.
Does it depend on the chosen input and output?	Yes. Different choices of input or output can produce different transfer functions for the same physical plant.
Does it contain the initial condition?	No. Initial-condition effects are handled separately from the transfer function.
Does it describe internal state uniquely?	No. It is an input–output description, not a complete internal-state description.
Can it be used for nonlinear or general time-varying models in the same way?	No. The standard rational transfer-function framework developed here is for LTI systems.
Why is it useful?	It converts dynamic input–output relations into algebraic functions of s that are easy to interconnect and analyze.

14 Main Takeaways

1. The Laplace transform converts linear constant-coefficient ODEs into algebraic equations in the complex variable s.
2. Derivatives become powers of s, while initial conditions appear explicitly as algebraic terms.
3. Inverse Laplace transforms and partial-fraction expansion provide a systematic way to recover time-domain responses.
4. For a specified input–output pair, the transfer function is the ratio $G(s) = Y(s)/U(s)$ under zero system initial conditions; it is not the pointwise ratio $y(t)/u(t)$.
5. A transfer function describes zero-state input–output behavior; it does not by itself encode the response caused by a nonzero initial state or provide a unique internal-state description.
6. The same physical plant can have different transfer functions for different choices of input and output.
7. After common factors have been removed, the roots of the numerator are the zeros and the roots of the denominator are the poles. For a proper transfer function with no pole–zero cancellation, the denominator degree gives the system order.
8. When $G(0)$ is finite, the DC gain gives the static output per unit constant input; for a stable system, it also determines the steady response to a constant input.

The central transition of this lecture is therefore

$$\boxed{\text{Physical System} \longrightarrow \text{ODE Model} \longrightarrow s\text{-Domain Algebra} \longrightarrow \text{Transfer Function.}}$$

Homework

1. Assume the following signals are zero for $t < 0$. Use linearity and the transform pairs in Table 1 to find their Laplace transforms:
   (a) $4 + 3t - 2e^{-5t}$;
   (b) $2\sin(3t) + 5\cos(3t)$.

2. Suppose $\mathcal{L}\{f(t)\} = F(s)$, $f(0^-) = 2$, and $\dot{f}(0^-) = -1$. Express

$$\mathcal{L}\{3\ddot{f}(t) + 4\dot{f}(t) + 2f(t)\}$$

   in terms of $F(s)$ and s.

3. Find the inverse Laplace transform for $t \geq 0$:
   (a) $F(s) = \dfrac{5}{s} - \dfrac{2}{s+4}$;
   (b) $F(s) = \dfrac{3s+8}{s(s+2)}$.

4. Consider the initial-value problem

$$2\dot{y}(t) + y(t) = u(t), \qquad y(0^-) = 1.$$

   The input changes from $u(t) = 0$ for $t < 0$ to $u(t) = 3$ for $t \geq 0$.
   (a) Use the Laplace transform to find $Y(s)$ and $y(t)$ for $t \geq 0$.

(b) Separate the response into its zero-state and zero-input components.
(c) Identify the steady component, the transient component, and the time constant.

5. Why must initial conditions be included when solving an ODE with the Laplace transform but set to zero when defining a transfer function? Explain what information would be lost if the initial conditions were discarded at the beginning of an initial-value problem.

6. For the mass–spring–damper model

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t),$$

derive, under zero initial conditions, the transfer functions from applied force to
(a) position $q(t)$;
(b) velocity $v(t) = \dot{q}(t)$;
(c) acceleration $a(t) = \ddot{q}(t)$.
Explain why the three transfer functions have the same denominator but different numerators.

7. For $m > 0$, $b > 0$, and $k > 0$, consider

$$G_q(s) = \frac{1}{ms^2 + bs + k}.$$

(a) Identify the system order, poles, zeros, and DC gain.
(b) Explain the physical meaning of the DC gain.
(c) Determine the condition on m, b, and k for the poles to be complex. What qualitative behavior would you then expect from the transient response?
(d) Describe qualitatively how increasing b affects the pole locations and the transient response.

8. Consider the linear input–output equation

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 2\dot{u}(t) + 5u(t).$$

Assume zero system initial conditions and $u(t) = 0$ for $t < 0$.
(a) Derive the transfer function $G_1(s) = Y(s)/U(s)$.
(b) Identify its order, poles, zeros, and DC gain, and determine whether it is proper.
(c) The system is connected in series with

$$G_2(s) = \frac{1}{s+4}.$$

Derive the equivalent transfer function from the original input to the final output, and identify its poles and DC gain.