ME 41100: System Dynamics and Control
Lecture 2: Modeling Dynamic Systems for Control

Bo Wang

Department of Mechanical Engineering, The City College of New York

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Learning Objectives. By the end of this lecture, you should be able to:

- explain what a dynamic model is and why useful models require simplifying assumptions;
- distinguish physics-based modeling from experimental or data-based modeling;
- define the system boundary and classify the reference, control input, output, disturbance, state, and parameters;
- apply physical laws and constitutive relations to derive simple translational mechanical models and extend the reasoning to rotational systems;
- identify the order of a dynamic system and the initial conditions required to determine its response;
- distinguish linear from nonlinear models and time-invariant from time-varying models;
- convert a second-order differential equation into an equivalent set of first-order state equations.

1 From a Physical System to a Dynamic Model

In Lecture 1, we used the mass–spring–damper model

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = u(t)$$

to illustrate how feedback reshapes system dynamics. We now ask a more fundamental question:

> **How do we turn a physical system into a mathematical model that is useful for control?**

This lecture focuses on the first step of the course workflow:

$$\boxed{\text{Physical System} \longrightarrow \text{Dynamic Model}} \longrightarrow \text{Represent} \longrightarrow \text{Analyze} \longrightarrow \text{Design} \longrightarrow \text{Verify.}$$

A practical modeling process can be organized as

$$\text{Physical system} \longrightarrow \text{Assumptions} \longrightarrow \text{Variables} \longrightarrow \text{Physical laws} \longrightarrow \text{Differential equations.}$$

2 What Is a Model?

A **model** is a mathematical description of the aspects of a physical system that are relevant to the engineering question of interest. A model is therefore a purposeful simplification of reality, not a complete copy of it.

> **The most detailed model is not necessarily the most useful model.**

Consider a vehicle used for cruise control. A simple longitudinal model may be written as

$$m\dot{v}(t) + Bv(t) = u(t) + d(t), \tag{1}$$

where v is vehicle speed, u is the commanded longitudinal force, Bv is a linear approximation of the speed-dependent resistive force, and d is a signed disturbance force representing effects such as road grade or wind. This model ignores suspension motion, tire deformation, engine combustion dynamics, structural vibration, and many other physical effects.

Those effects are real, but they may not be important for the control objective and time scale of interest. A model should therefore be chosen according to

- the **objective**: what behavior are we trying to predict or control?
- the **operating range**: where will the system be used?
- the **relevant time scale**: which dynamics are fast or slow relative to the behavior of interest?
- the **required accuracy**: how much modeling error can the analysis or design tolerate?

2.1 Physics-Based and Data-Based Modeling

Models can be derived from physical laws, identified from experimental data, or constructed using a combination of both:

$\boxed{\text{Physics-Based (First-Principles) Modeling}}$ and/or $\boxed{\text{Experimental/Data-Based Modeling}}$

In ME 41100, we will primarily use *physics-based models*. Experiments will also be used to estimate parameters and to check whether a model captures the observed behavior over the operating conditions of interest.

Question for Discussion

If the same vehicle were being studied for ride comfort rather than cruise control, which effects omitted from Equation (1) might become important?

3 Anatomy of a Control-Oriented Dynamic Model

A general dynamic model can be written schematically as

$$\dot{\mathbf{x}}(t) = f\big(\mathbf{x}(t), u(t), d(t); \theta\big), \tag{2}$$

$$y(t) = h\big(\mathbf{x}(t), u(t), d(t); \theta\big), \tag{3}$$

where the notation separates quantities according to the role they play in the control problem. Inputs, disturbances, and outputs may be scalars or vectors; most early examples in this course will be single-input, single-output (SISO) systems. The reference $r(t)$ does not appear in the plant equations because it specifies the desired behavior rather than the physical evolution of the plant. It enters the model when the controller is included.

Table 1: Basic quantities in a control problem and its dynamic model.

Quantity	Symbol	Meaning
State	$\mathbf{x}$	Internal variables needed to describe the current condition of the system and predict its future evolution.
Reference	r	The desired value or trajectory for the controlled output; it is generally external to the physical plant model.
Control input	u	A quantity that the controller is allowed to manipulate.
Disturbance	d	A quantity that affects the system but is not commanded by the controller.
Output	y	A quantity that is measured or is of interest for the control objective.
Parameters	θ	Physical constants such as mass, inertia, damping, stiffness, geometry, or actuator coefficients.

Before writing equations, we must also choose a **system boundary**: what is included in the model, and what is treated as an external input or disturbance? A signal can be an output of one subsystem and an input to another, so the roles of inputs and outputs depend on the chosen boundary and point of view.

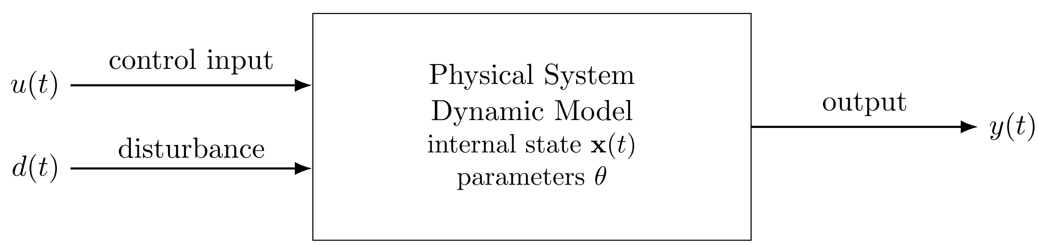

Figure 1: The system boundary determines which quantities are modeled internally and which enter from outside.

Question for Discussion

For automobile cruise control, classify the following quantities: vehicle speed, commanded longitudinal force, road-grade disturbance force, vehicle mass, and desired speed. Which quantities belong inside the physical plant model, and which cross its boundary?

4 Mechanical Modeling Elements

For mechanical systems, dynamic models follow from physical laws together with constitutive relations for individual components.

Mass and Rotational Inertia. Newton's second law relates net force to translational acceleration:

$$\sum F = m\ddot{x}. \tag{4}$$

The rotational analogue is

$$\sum \tau = J\ddot{\theta}, \tag{5}$$

where J is the rotational inertia (mass moment of inertia).

Linear Spring. For a linear spring with one end fixed and x measured from equilibrium, the force exerted on the moving body is

$$F_s = -kx. \tag{6}$$

The negative sign indicates that the spring force opposes the displacement. The spring stores elastic potential energy.

Viscous Damper. For an ideal viscous damper with one end fixed, the force exerted on the moving body opposes its motion and is proportional to its velocity:

$$F_b = -b\dot{x}. \tag{7}$$

The damper dissipates mechanical energy.

Spring and damper forces depend on the *relative* displacement and relative velocity of their two endpoints. Their rotational counterparts obey analogous relations in angle and angular velocity.

Table 2: Basic translational and rotational mechanical elements.

Element	**Constitutive relation**	**Physical role**
Translational mass	$F = m\ddot{x}$	inertia / kinetic-energy storage
Rotational inertia	$\tau = J\ddot{\theta}$	inertia / kinetic-energy storage
Linear spring	$F_s = -kx$	elastic-energy storage
Torsional spring	$\tau_s = -k_\theta \theta$	elastic-energy storage
Viscous damper	$F_b = -b\dot{x}$	energy dissipation
Rotational damper	$\tau_b = -b_\theta \dot{\theta}$	energy dissipation

Question for Discussion

Which ideal elements store energy, and which dissipate it? How might the amount and type of dissipation affect transient response and stability?

5 Example: Modeling a Mass–Spring–Damper System

Consider the system in Figure 2. Let $q(t)$ denote the displacement of the mass from its equilibrium position. The applied control force is $u(t)$, and an external disturbance force $d(t)$ may also act on the mass.

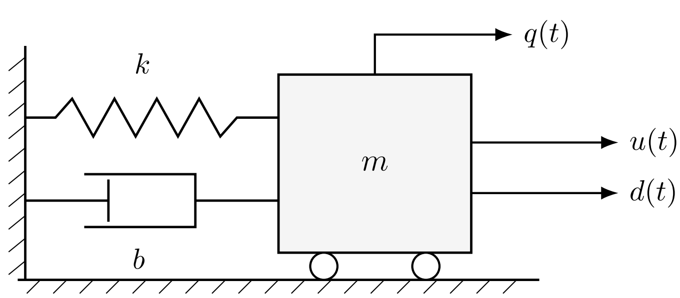

Figure 2: Mass–spring–damper system used to illustrate the modeling process.

Modeling Assumptions

Before deriving the governing equation, we state the main assumptions:

- the moving body is rigid and is represented by a lumped mass m, with horizontal translation as its only motion;
- the spring is linear with stiffness k;
- the damper is viscous with coefficient b;
- the parameters m, k, and b are constant;
- Coulomb friction, backlash, structural flexibility, and geometric nonlinearities are neglected;
- actuator and sensor dynamics are not included in the plant model.

Step 1: Choose the Coordinate and Sign Convention

Define $q = 0$ at the equilibrium position and take displacement to the right as positive. Thus, $q > 0$ gives a leftward spring force, and $\dot{q} > 0$ gives a leftward damping force. The arrows for $u(t)$ and $d(t)$ indicate their positive directions; either force may take a negative value.

Step 2: Draw the Free-Body Diagram

The forces acting on the mass in the horizontal direction are $u(t)$, $d(t)$, $-kq(t)$, and $-b\dot{q}(t)$. Figure 3 shows their directions when $u, d, q, \dot{q} > 0$.

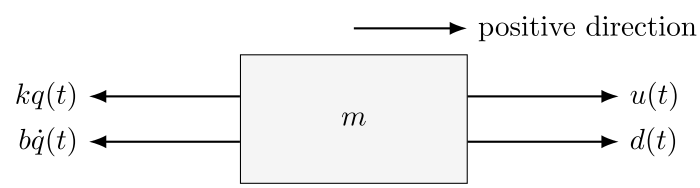

Figure 3: Free-body diagram for positive u, d, q, and $\dot{q}$.

Step 3: Apply Newton's Second Law

Summing forces in the positive direction gives

$$m\ddot{q}(t) = u(t) + d(t) - b\dot{q}(t) - kq(t),$$

so the governing differential equation is

$$\boxed{m\ddot{q}(t) + b\dot{q}(t) + kq(t) = u(t) + d(t).} \tag{8}$$

Step 4: Identify the Model Quantities

For this model,

- the **control input** is $u(t)$;
- the **disturbance** is $d(t)$;
- the **parameters** are m, b, and k;
- a convenient **state vector** is $\mathbf{x}(t) = \begin{bmatrix} q(t) & \dot{q}(t) \end{bmatrix}^\top$;

- the **output** is selected according to the sensing arrangement and control objective: $y = q$, $y = \dot{q}$, or $\mathbf{y} = \begin{bmatrix} q & \dot{q} \end{bmatrix}^\top$.

Question for Discussion

Suppose the position sensor is replaced by a velocity sensor. Does Equation (8) change? What changes in the mathematical description of the control system?

6 When Assumptions Change the Mathematics

Equation (8) is useful because it captures the dominant motion with a small number of variables and parameters. It is not an exact description of every physical effect. Changing an assumption can change both the equation and the mathematical tools needed to analyze it.

For example, linear viscous damping and idealized Coulomb friction can be compared as

$$F_v = -b\dot{q}, \qquad \text{versus} \qquad F_{\mathrm{C}} = -F_0 \operatorname{sgn}(\dot{q}), \quad \dot{q} \neq 0.$$

The viscous-damping force is linear and its magnitude grows with speed. The Coulomb-friction force has constant magnitude at nonzero velocity, opposes the direction of motion, and introduces a nonlinear and nonsmooth effect. At zero velocity, static friction requires a more detailed model.

Similarly, a nonlinear spring might satisfy

$$F_s = -k_1 q - k_3 q^3. \tag{9}$$

Substituting this force law into Newton's second law gives

$$m\ddot{q} + b\dot{q} + k_1 q + k_3 q^3 = u + d. \tag{10}$$

The cubic spring produces a smooth nonlinear model, whereas idealized Coulomb friction produces a nonsmooth nonlinear model.

A model is judged by whether it captures the behavior needed for analysis and design over the operating conditions of interest—not by whether it includes every physical detail.

7 Basic Structure of Dynamic Models

Before developing transfer functions, we briefly introduce three basic ways to classify differential-equation models.

System Order and Initial Conditions. The **order** of an ordinary differential equation is the order of its highest derivative. For example,

$$T\dot{y}(t) + y(t) = Ku(t) \tag{11}$$

is a first-order model, where $T > 0$ is a time constant, while Equation (8) is second order.

A differential equation alone does not determine a unique trajectory. Initial conditions are also required. A first-order model requires one initial condition, such as $y(0)$; a second-order mechanical model requires two, such as $q(0)$ and $\dot{q}(0)$. In general, a scalar nth-order ODE requires n independent initial conditions.

Linear and Nonlinear Models. A differential-equation model is **linear** if the dependent variables, their derivatives, and the inputs appear linearly. Equivalently, with zero initial conditions, its input–output map satisfies superposition. A linear mechanical model is

$$m\ddot{q} + b\dot{q} + kq = u.$$

Nonlinear models contain terms that violate superposition, for example

$$m\ddot{q} + b\dot{q} + k_1 q + k_3 q^3 = u$$

or the nonlinear pendulum model

$$J\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u.$$

Here u denotes the applied torque.

Linear models are especially important because they admit powerful analysis and design tools and often provide useful approximations to nonlinear systems near an operating condition.

Time-Invariant and Time-Varying Models. A model is **time invariant** if its governing relationships do not change explicitly with time. For example,

$$m\ddot{q} + b\dot{q} + kq = u$$

is time invariant when m, b, and k are constants. A model such as

$$m(t)\ddot{q} + b(t)\dot{q} + k(t)q = u$$

is time varying because its coefficients change explicitly with time.

Time invariance does not mean that the input or response is constant; it means that the governing law is unchanged when the experiment is shifted in time. Linearity and time invariance are separate properties: the coefficient-varying model above is linear but time varying, whereas the nonlinear pendulum with constant parameters is nonlinear but time invariant.

Much of ME 41100 will focus on **linear time-invariant (LTI)** systems. These models are simple enough to analyze systematically while still describing a large class of engineering systems and useful local approximations.

8 What Is a State?

The **state** is a set of variables that, together with the model and future inputs, determines the future evolution of a dynamic system. It summarizes all information from the past that is relevant for predicting the future.

For the mass–spring–damper system, set $d(t) = 0$ for simplicity and define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \tag{12}$$

Then

$$\dot{x}_1(t) = x_2(t), \tag{13}$$

$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{b}{m}x_2(t) + \frac{1}{m}u(t). \tag{14}$$

Thus one second-order equation has been rewritten as two coupled first-order equations.

If position is measured, the output equation is $y = x_1$; if velocity is measured, it is $y = x_2$. Changing the measured output does not change the state equations.

A scalar nth-order ODE can be rewritten as n coupled first-order state equations, or equivalently as one n-dimensional state equation.

The choice of state variables is generally not unique. Different independent coordinates can represent the same internal condition, provided that the chosen state and the future input are sufficient to determine the future evolution.

At this point we will use state variables mainly as a modeling concept. Later in the course, state-space methods will provide a systematic framework for analysis, feedback design, and estimation.

9 Different Physics, Same Mathematical Structure

The same differential-equation structure appears in many physical systems. Consider a rotational inertia connected to a torsional spring and viscous rotational damper:

$$J\ddot{\theta}(t) + b_\theta\dot{\theta}(t) + k_\theta\theta(t) = \tau(t). \tag{15}$$

Compare this with the translational model

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = F(t).$$

An example from a different physical domain is a series RLC circuit. If $q_e(t)$ is the capacitor charge and $v(t)$ is the applied voltage, Kirchhoff's voltage law gives

$$L\ddot{q}_e(t) + R\dot{q}_e(t) + \frac{1}{C}q_e(t) = v(t).$$

Table 3: Analogous mathematical roles in three physical domains.

Mathematical role	**Translation**	**Rotation**	**Series RLC**
Coordinate	q	θ	q_e
Second-derivative coefficient	m	J	L
First-derivative coefficient	b	b_θ	R
Coordinate coefficient	k	k_θ	$1/C$
Input	F	τ	v

The corresponding quantities play the same mathematical roles, although their physical meanings and units are different.

Different physical systems can share the same mathematical dynamic structure.

This is one reason system dynamics and control methods are broadly reusable: once a mathematical structure is understood, the same analysis tools can often be applied to many different physical systems.

10 A Practical Modeling Workflow

When approaching a new control problem, the following sequence is useful:

1. **Define the objective and system boundary.** What behavior matters, and what is inside the model?
2. **State modeling assumptions.** Which physical effects will be retained or neglected?
3. **Choose coordinates and sign conventions.** What variables describe the motion?
4. **Identify inputs, disturbances, outputs, states, and parameters.**
5. **Apply physical laws.** Use force and torque balances, energy methods, circuit laws, or conservation laws as appropriate.
6. **Obtain and check the differential equations.** Check units, signs, limiting cases, and physical interpretation.
7. **Validate and refine the model.** When data are available, compare predicted and measured behavior; add complexity only when the simpler model cannot explain the behavior relevant to the engineering objective.

Question for Discussion

What is the simplest check you can perform immediately after deriving a differential equation? What kinds of modeling errors can dimensional consistency and limiting cases reveal?

11 Main Takeaways

1. Modeling begins by defining the system boundary and the engineering objective.
2. References, control inputs, disturbances, outputs, states, and parameters play distinct roles in a control problem.
3. Models can be derived from physical laws, identified from experimental data, or constructed using both approaches.
4. Physical laws and constitutive relations lead to differential equations describing system evolution.
5. Every model contains assumptions; the appropriate model is the one that captures the dynamics relevant to the analysis or control objective.
6. System order determines how many independent initial conditions—and state variables—are needed.
7. Different physical systems can share the same mathematical structure, which allows common analysis and control tools to be reused.

The next step in the course workflow is to develop mathematical representations that make dynamic models easier to analyze. We will begin with the Laplace transform and transfer functions.

Homework

Reading: Read Chapter 3 of *Feedback Systems: An Introduction for Scientists and Engineers*, then answer the questions below.

1. A rotational inertia J is connected to ground through a torsional spring of stiffness k_θ and a viscous rotational damper with coefficient b_θ. An applied torque $\tau(t)$ acts on the inertia. Draw the free-body diagram and derive the governing differential equation for the angular displacement $\theta(t)$.
2. For the mass–spring–damper system in Equation (8), suppose that velocity rather than position is measured. Write the governing differential equation and the output equation. Does changing the measured output change the plant dynamics?
3. Classify each model as linear or nonlinear and as time-invariant or time-varying:
   (a) $m\ddot{x} + b\dot{x} + kx = u$;
   (b) $m\ddot{x} + b\dot{x} + kx^3 = u$;
   (c) $m\ddot{x} + b(t)\dot{x} + kx = u$;
   (d) $J\ddot{\theta} + b_\theta\dot{\theta} + mgl\sin\theta = \tau$.
4. Convert the second-order equation

$$m\ddot{q} + b\dot{q} + kq = u$$

   into two first-order equations using $x_1 = q$ and $x_2 = \dot{q}$.
5. Give one example of a physical effect that you might intentionally neglect when building a control model. Explain what engineering objective or operating condition would justify neglecting it.