Safety-Critical Stabilization of Force-Controlled Nonholonomic Robots

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Abstract

We present a *safety-critical controller* for the *stabilization* problem for force-controlled nonholonomic vehicles.

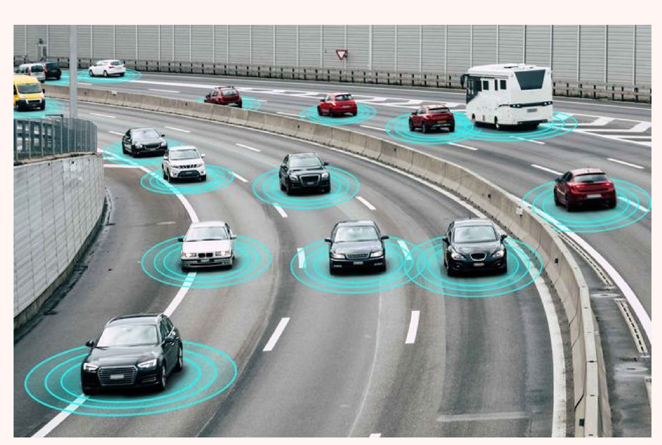
- The control law is based on the constructions of control Lyapunov functions (CLFs) and control barrier functions (CBFs) for cascaded systems.
- Quadratic programming (QP) is employed to combine CLFs and CBFs to integrate both stability and safety in the closed loop.

The control law is *time-invariant* and *continuous* along trajectories.

Motivation & Objective

Ensuring *safety* is crucial in vehicle control applications. Autonomous systems must satisfy *strict safety requirements*, including the avoidance of obstacles and inter-vehicle collisions.

- **Parking with obstacle avoidance:** A vehicle parks in a designated space while actively avoiding obstacles in its path.
- **Autonomous Driving:** Vehicles navigate and operate in traffic while avoiding accidents with other vehicles.



Constructions of CBFs for Cascaded Systems

- Consider the admission set $\mathcal{C} \subset \mathbb{R}^n$ and the cascaded system

$$\dot{x}_1 = f(x_1) + g(x_1)x_2$$
$$\dot{x}_2 = u$$

- Suppose that we know a CBF $B_1 : \operatorname{int}\mathcal{C} \to \mathbb{R}_{>0}$ for the system $\dot{x} = f(x) + g(x)u$ and a "virtual" controller $x_2^* : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$L_{f_{\mathsf{safe}}} B_1(x_1) = L_f B_1(x_1) + L_g B_1(x_1) x_2^*(x_1) < \alpha_B(1/B_1(x_1))$$

for some $\alpha_B \in \mathcal{K}$ and for all $x_1 \in \operatorname{int}\mathcal{C}$. With $\tilde{x}_2 := x_2 - x_2^*(x_1)$, the cascaded system becomes

$$\dot{x}_1 = f_{\mathsf{safe}}(x_1) + g(x_1)\tilde{x}_2$$
$$\dot{\tilde{x}}_2 = u - \dot{x}_2^* =: \tilde{u}.$$

- Then the function $B : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{>0}$ defined by

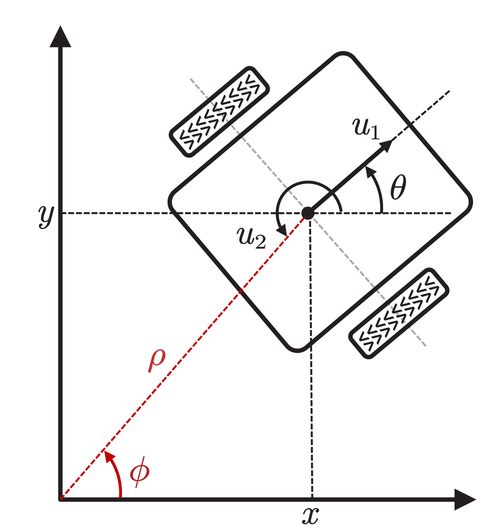
$$B(\mathbf{x}) := B_1(x_1) + \tilde{x}_2^\top H \tilde{x}_2, \quad H = H^\top > 0$$

is a CBF for the cascaded system, where $\mathbf{x} := [x_1\ \tilde{x}_2]^\top$.

Main Result: Controller Design

Model of Mobile Robots in Polar Coordinates

By defining $\alpha := \theta - \phi$, the model of mobile robots can be expressed in the polar coordinate.



kinematics: $$\begin{cases} \dot{\rho} = v\cos\alpha \\ \dot{\phi} = \dfrac{v}{\rho}\sin\alpha \\ \dot{\alpha} = \omega - \dfrac{v}{\rho}\sin\alpha. \end{cases}$$

kinetics: $$\begin{cases} \dot{v} = u_1 \\ \dot{\omega} = u_2 \end{cases}$$

CLF Construction in Polar Coordinates

Following standard backstepping, let $(v, \omega) := (v^*, \omega^*) + (\tilde{v}, \tilde{\omega})$. We design virtual control laws (v^*, ω^*) to stabilize the kinematics, *i.e.*,

$$v^* = -k_\rho \cos(\alpha)\rho$$
$$\omega^* = -k_\alpha \alpha - k_\rho \operatorname{sinc}(2\alpha)(\alpha - \lambda\phi)$$

then use (u_1, u_2) to stabilize the kinetics in $(z := \tilde{v}/\rho, \tilde{\omega})$ coords., *i.e.*,

$$u_1 = \dot{v}^* - \rho(k_\rho \cos(\alpha)^2 z - \cos(\alpha) z^2 + k_z z)$$
$$u_2 = \dot{\omega}^* - k_\omega \tilde{\omega}$$

The closed-loop system is in a cascaded structure:

$$\begin{bmatrix} \dot{\rho} \\ \dot{\phi} \\ \dot{\alpha} \end{bmatrix} = \underbrace{\begin{bmatrix} -k_\rho \cos(\alpha)^2 \rho \\ -k_\rho \operatorname{sinc}(2\alpha)\alpha \\ -k_\alpha \alpha + \lambda k_\rho \operatorname{sinc}(2\alpha)\phi \end{bmatrix}}_{f(\rho,\phi,\alpha)} + \underbrace{\begin{bmatrix} \rho\cos\alpha & 0 \\ \sin\alpha & 0 \\ -\sin\alpha & 1 \end{bmatrix}}_{g(\rho,\alpha)} \begin{bmatrix} z \\ \tilde{\omega} \end{bmatrix}$$

$$\dot{z} = -k_z z$$
$$\dot{\tilde{\omega}} = -k_\omega \tilde{\omega}.$$

The strict Lyapunov function is given by

$$\mathcal{V}_r(\rho, \phi, \alpha, z, \tilde{\omega}) := \mu \ln(V(\rho, \phi, \alpha) + 1) + U(z, \tilde{\omega})$$

where

$$V(\rho, \phi, \alpha) := \frac{\nu(r)}{2}(\rho^2 + \lambda\phi^2 + \alpha^2) + \xi^\top P \xi$$

$$U(z, \tilde{\omega}) := \frac{1}{2}\left(\frac{z^2}{k_z} + \frac{\tilde{\omega}^2}{k_\omega}\right)$$

CBF Construction in Cartesian Coordinates

Assume that the admissible set $\mathcal{C} := \{(x, y) \in \mathbb{R}^2 : h(x, y) \geq 0\}$ is given, where $h : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is continuously differentiable.

For the mobile robot in Cartesian coords., $B(\mathbf{x}) := 1/h(x, y) + \eta^\top H \eta$ is a CBF, where $\mathbf{x} := [x\ y\ v\ \omega]^\top$, $\eta := [v\ \omega]^\top$, and $H = H^\top > 0$.

Safety-Critical Control Design

For any $r > 0$, the γm-QP problem is defined as

$$\min \frac{1}{2}(\bar{u}^\top \bar{u} + m\delta^\top \delta)$$
$$\text{s.t. } F_1 := \gamma_f(L_{f_1}\mathcal{V}_r + \alpha(|\chi|)) + L_{g_1}\mathcal{V}_r \bar{u} + L_{g_1}\mathcal{V}_r \delta \leq 0$$
$$F_2 := L_{f_2}B(\mathbf{x}) - \alpha_B(1/B(\mathbf{x})) + L_{g_2}B(\mathbf{x})\bar{u} \leq 0$$

where $\chi := [\rho\ \phi\ \alpha\ z\ \tilde{\omega}]^\top$, $\alpha_B \in \mathcal{K}$, $\alpha := \frac{\epsilon L_f V}{V+1} - \frac{1}{2}|\zeta|^2$, and $\epsilon > 0$ is chosen to be sufficiently small. The closed-form solution of the γm-QP problem can be obtained by invoking the KKT conditions.

Simulation Results

We present two simulation examples to illustrate the performance of the proposed controller. In each example, the proposed CLF-CBF γm-QP controller is compared with the nominal controller and the CLF-based pointwise minimum norm (PMN) controller.

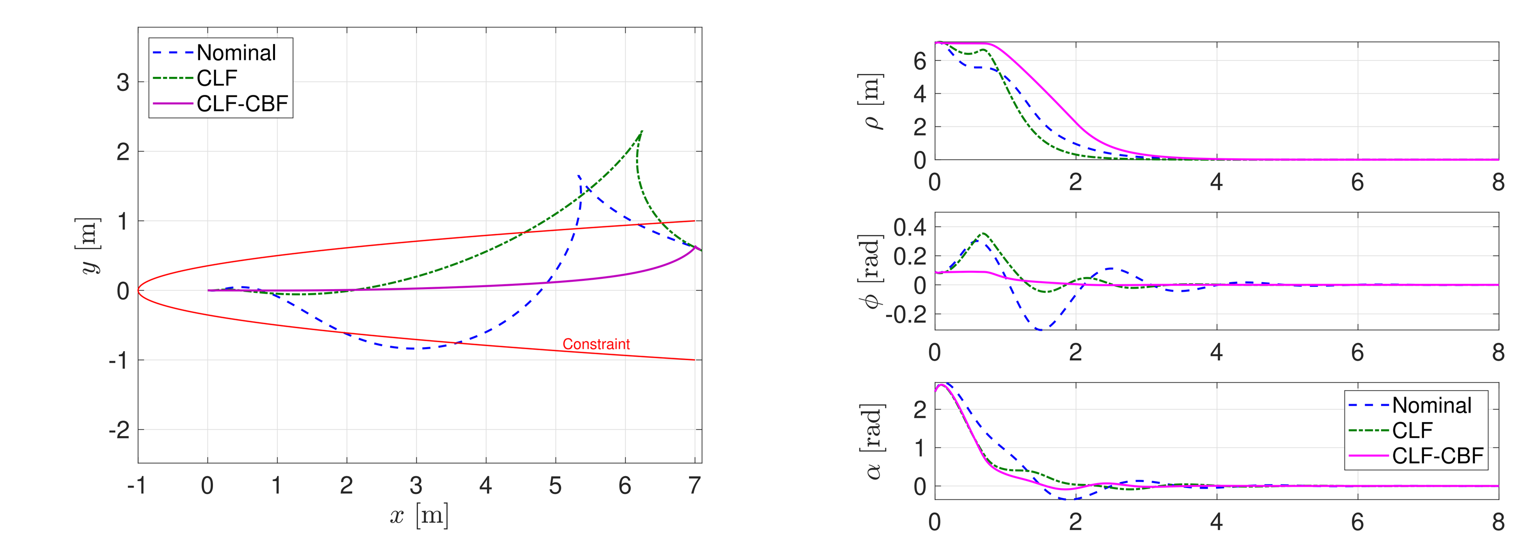


Figure 2. Comparison of the robot paths and convergence of variables in Example 1.

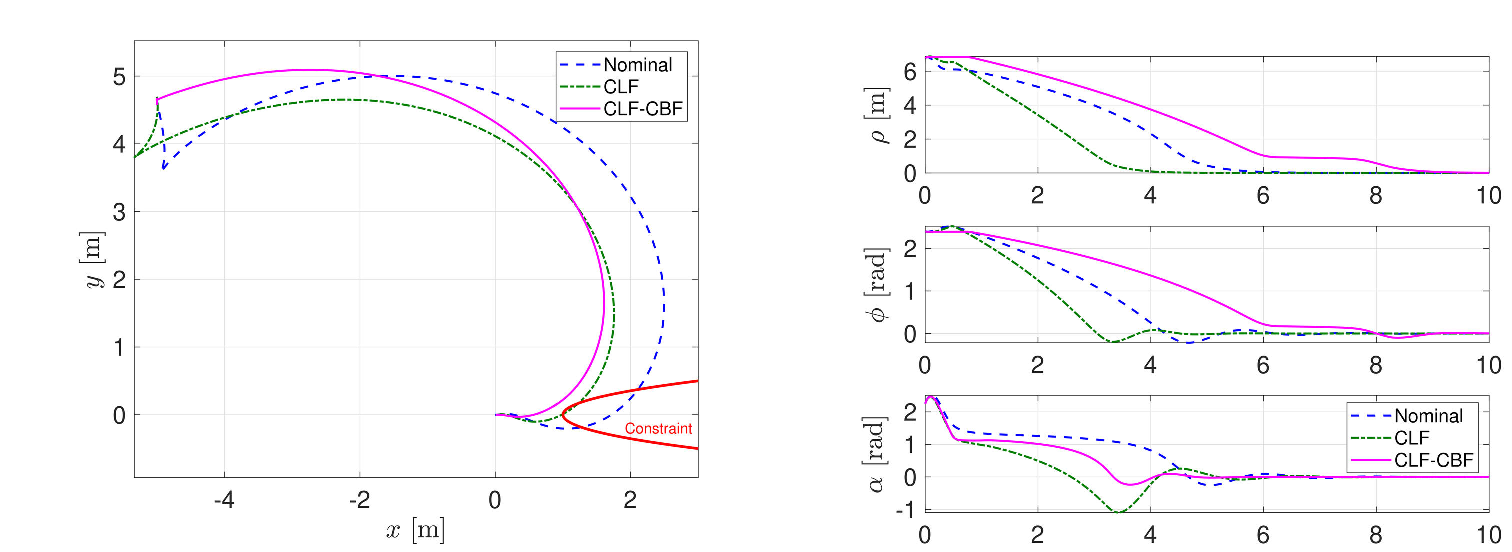


Figure 3. Comparison of the robot paths and convergence of variables in Example 2.

The simulation results are shown in Figs. 2 and 3, which demonstrate that the proposed CLF-CBF γm-QP controller effectively achieves *parking with obstacle avoidance*.

References

[1] T. Han, B. Wang, "Safety-Critical Stabilization of Force-Controlled Nonholonomic Robots," *arXiv preprint*, arXiv:2408.10941, August 2024.

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